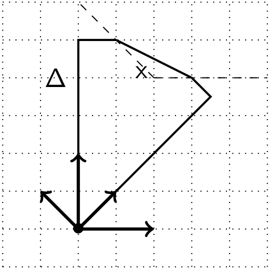
Existence of canonical Kähler metrics on spherical varieties — Lecture 3

Ensemble of Algebra and Geometry



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cscK metrics

ω Kähler form on X

Scalar curvature function:

$$S(\omega) = \frac{n\,\mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n} : X \to \mathbb{R}$$

A Kähler form is cscK (constant scalar curvature Kähler) if

$$S(\omega) = \bar{S}$$

for some $\bar{S} \in \mathbb{R}$
e.g. if ω is Kähler-Einstein with $\mathrm{Ric}(\omega) = t\omega$, then ω is cscK with

$$S(\omega) = \frac{n(t\omega) \wedge \omega^{n-1}}{\omega^n} = nt$$

One usually search for a cscK metric in a given Kähler class α.
Then

$$\bar{S} = \frac{nc_1(X) \cdot \alpha^{n-1}}{\alpha^n}$$

is a cohomological constant.

More generally

Extremal Kähler metric in the Kähler class α in the sense of Calabi = minimizer of

$$\alpha \to \mathbb{R}, \omega \mapsto \int_X S(\omega)^2 \omega^n$$

e.g. cscK metrics

This turns out to be equivalent to: $S(\omega)$ is the potential function of a holomorphic vector field

Two unrelated advantages:

- second definition does not require to fix the class
- if the class is fixed, can determine the holomorphic vector field *a priori* using an adapted notion of Futaki invariant.

Test configurations

(X, L) polarized variety

A test configuration for (X, L) consists of the data of

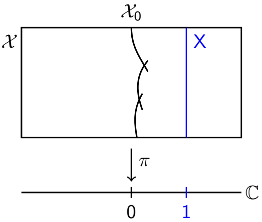
1. a normal variety $\mathcal{X}$,
2. a $\mathbb{C}^*$-action on $\mathcal{X}$,
3. a flat projective, $\mathbb{C}^*$-equivariant morphism $\pi : \mathcal{X} \to \mathbb{C}$,
4. a π-ample line bundle $\mathcal{L}$ on $\mathcal{X}$,

such that

1. $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L^r)$ for some $r \in \mathbb{Z}_{>0}$,

where $(\mathcal{X}_1, \mathcal{L}_1)$ denotes the (scheme-theoretic) fiber of π above $1 \in \mathbb{C}$, equipped with the restriction of $\mathcal{L}$.

$(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}) \simeq (X \times \mathbb{C}^*, L^r)$



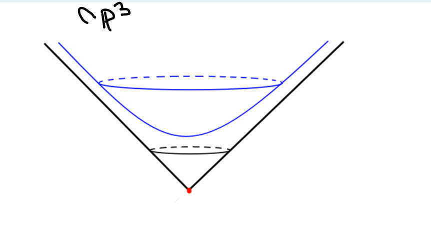
$\pi^{-1}(1) = X_1 \simeq X$

The central fiber $(\mathcal{X}_0, \mathcal{L}_0)$, equipped with an action of $\mathbb{C}^*$, is the important data.
Upshot: more symmetries, may acquire singularities, e.g. non-reduced, several irreducible components, other singularities.
Can also think of $\mathcal{X}_0$ as a $\mathbb{C}^*$-stable divisor in $\mathcal{X}$.

A family of examples

Degeneration of a quadric to the cone over a lower-dimensional quadric.

- $\mathbb{P}^1$ degenerates to two intersecting lines (several irreducible components)
  $\mathcal{X} = \{([x : y : z], t); xy - tz^2 = 0\}$
- $\mathbb{P}^1$ degenerates to a double line (non-reduced)
  $\mathcal{X} = \{([x : y : z], t); txy - z^2 = 0\}$
- $\mathbb{P}^1 \times \mathbb{P}^1$ degenerates to a weighted projective space (normal, but singular)



Special test configurations

Special test configuration: A test configuration is called *special* if the central fiber $\mathcal{X}_0$ is normal (in particular, reduced and irreducible, but may still be singular, see the case of quadrics).
Note:

- [Li-Xu 2011]: for Fano manifolds (varieties), enough to consider special test configurations
  K-stability $\iff$ K-stability for special t.c.
- [Donaldson 2002]: for cscK metrics on toric surfaces, special test configurations are not enough.

Product test configuration: A test configuration is called *product* if $\mathcal{X}_0$ is isomorphic to X.
Any $\mathbb{C}^*$-action on (X, L) generates a product test configuration as follows: Consider $\mathcal{X} = X \times \mathbb{C}$, π the projection to $\mathbb{C}$, and the action of $\mathbb{C}^*$ given by $t \cdot (x, s) = (t \cdot x, ts)$. All product test configurations arise from this construction.

$\mathcal{L} = p_1^* L \quad \mathcal{X} \xrightarrow{p_1} X$

Donaldson-Futaki invariant

$$H^0(\mathcal{X}_0, \mathcal{L}_0^k) = \bigoplus_{j=1}^{d_k} V_{j,k} \text{ decomposition in irreducible } \mathbb{C}^*\text{-representations}$$

Each $V_{j,k}$ is of dimension one, and $\mathbb{C}^*$ acts by on it with a weight $\lambda_{j,k}$ $\in \mathbb{Z}$:

$$\forall s \in V_{j,k}, \quad z \cdot s = z^{\lambda_{j,k}} s$$

$$\frac{\sum_j \lambda_{j,k}}{kd_k} = F_0 + F_1 \frac{1}{k} + o(\frac{1}{k}) \text{ as } k \to \infty \text{[Donaldson 2002]}$$

Donaldson-Futaki invariant $DF(\mathcal{X}, \mathcal{L}) := -F_1$
Non-archimedean J-functional $J^{NA}(\mathcal{X}, \mathcal{L}) = \sup\{\frac{\lambda_{j,k}}{k}\} - F_0$
Subtleties:

- F_0 depends on the choice of $\mathbb{C}^*$-linearization of $\mathcal{L}$ (but not J^{NA})
- DF does not vary linearly with base changes $z \mapsto z^m$. Better to work with non-Archimedean Mabuchi functional M^{NA} which is the linear functional which coincide with DF when the central fiber is reduced.

(Uniform) K-stability

Note: For product test configurations, the Donaldson-Futaki invariant coincides with the Futaki invariant of the generator of the $\mathbb{C}^*$ action.

Definitions

- (X, L) is **K-semistable** if $DF(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations.
- (X, L) is **K-(poly)stable** if $DF(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations, with equality if and only if the test configuration is a product.
- (X, L) is **uniformly K-stable** if there exists a positive constant $\epsilon > 0$ such that for all test configurations, $M^{NA}(\mathcal{X}, \mathcal{L}) \geq \epsilon J^{NA}(\mathcal{X}, \mathcal{L})$.

In presence of a group action, can/must require that the test configurations respect the group action: the $\mathbb{C}^*$ action on $\mathcal{X}$ must commute with the given G-action on X. $G \times \mathbb{C}^* \curvearrowright \mathcal{X}$

[Datar-Szekelyhidi 2015] For smooth Fano manifolds, equipped with the action of a reductive group G, enough to consider G-equivariant special test configurations

[Chi Li 2021, based on Hisamoto and indirectly, Donaldson 2002] Notion of uniform K-stability allowing to take into account group actions, requires the notion of *twist* of a test configuration.

Twists of test configurations

Recall how product test configurations were defined in terms of the trivial test configurations. A twist of a test configuration is the same construction applied from an arbitrary test configuration.
Let $(\mathcal{X}, \mathcal{L})$ be a G-equivariant test configuration, and let $\mu : \mathbb{C}^* \to \mathrm{Aut}^G(X)$ be a one-parameter subgroup of the group of G-equivariant automorphisms of X.
Then, at least over $\mathbb{C}^*$, the family is trivial and one can define a new $\mathbb{C}^*$-action on $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times \mathbb{C}^*$ by $t \cdot (x, s) = (\mu(t) \cdot x, s)$. Actually, the action extends to $\mathcal{X}$ and defines a new G-equivariant test configuration, the **twist** of $(\mathcal{X}, \mathcal{L})$ by μ.

G-uniform K-stability

$\exists \epsilon > 0$, for all G-equivariant test configurations,

$$M^{NA}(\mathcal{X}, \mathcal{L}) \geq \epsilon \inf_{\mu} J^{NA}(\text{twist of } (\mathcal{X}, \mathcal{L}) \text{ by } \mu)$$

(actually, should allow $\mu \in X_*(\mathrm{Aut}^G(X)) \otimes \mathbb{R}$)

General YTD conjecture

Existence of a canonical Kähler metric in $c_1(L)$ should be equivalent to an algebro-geometric notion of stability for (X, L)
Partial results:

[Datar Szekelyhidi]

1. Kähler-Einstein metrics on Fano manifolds [Chen-Donaldson-Sun, Tian, 2015]
2. coupled g-solitons [Li-Han] [Berman Boucksom Jonsson]
3. existence cscK (extremal) $\Rightarrow$ uniform K-stability [Berman-Darvas-Lu 2020]
4. From the analytical point of view, [Chen-Cheng 2021] proved that coercivity (modulo automorphisms) of the Mabuchi functional implies existence of cscK metrics
5. [Chen-Cheng 2021 + Chi Li] some algebraic notion close to uniform K-stability $\Rightarrow$ existence cscK

What is the "best" notion of K-stability applicable?
(especially in direction K-stability $\Rightarrow$ existence)

The toric case

2002 2008 21

Theorem [Donaldson, Zhou-Zhu, Chen-Cheng 2018]

A polarized toric T-manifold (X, L) admits a cscK metric if and only if it is T-uniformly K-stable.

Better: translate into a convex geometric problem! [Donaldson]

Another proof was recently given by Chi Li (bypassing at least the [Zhou-Zhu] argument)

Theorem [Donaldson 2009]

A polarized toric T-manifold (X, L) of dimension 2 admits a cscK metric if and only if it is T-equivariantly K-stable.

Better: reduce to a finite dimensional space of conditions to check!

$$\mathcal{L}(g) = \int_{\partial\Delta} g\, d\sigma - \int_{\Delta} g\, 2a\, d\mu > 0$$

$g : \Delta \to \mathbb{R}$ convex PL not affine

enough to consider $g = \sup(0, ax + b)$

$a \in N \otimes \mathbb{R}$, $b \in \mathbb{R}$

Spherical case

Theorem [Odaka, appendix to one of my preprint]

A polarized G-spherical manifold (X, L) admits a cscK metric if and only if it is G-uniformly K-stable.

The result actually follows essentially from Chi Li's argument to prove the uniform YTD conjecture for toric manifolds. Odaka's observation is that it applies to spherical varieties as well.

Theorem [D]

Convex geometric translation of G-uniform K-stability for polarized G-spherical varieties.

Theorem [D]

A rank one polarized G-spherical manifold (X, L) admits a cscK metric if and only if it is K-stable with respect to G-equivariant **special** test configurations.
The latter translates into a very simple single combinatorial condition.

Combinatorial condition

Let (X, L) be a polarized rank one G-spherical manifold, with moment polytope Δ. Let $\chi \in \Delta$, let σ be a generator of M which evaluates non-negatively on the valuation cone, and write

$$\Delta = \{\chi + t\sigma \mid t \in [s_-, s_+]\}$$

$[s_-, s_+] \subset \mathbb{R}$

Let

$$P(t) = \prod_{\alpha \in \Phi_X^+} \frac{\langle \alpha, \chi + t\sigma \rangle}{\langle \alpha, \varpi \rangle} \qquad Q(t) = \sum_{\alpha \in \Phi_X^+} \frac{\langle \alpha, \varpi \rangle}{\langle \alpha, \chi + t\sigma \rangle} P(t)$$

For a continuous function $g : [s_-, s_+] \to \mathbb{R}$, let

$$\mathcal{L}(g) = g(s_-)P(s_-) + g(s_+)P(s_+) - \int_{s_-}^{s_+} 2g(t)(aP(t) - Q(t))dt$$

where a is the constant such that $\mathcal{L}(1) = 0$.
Then there exists a cscK metric iff

- $\mathcal{L}(\mathrm{id}) > 0$ if X is not horospherical,
- $\mathcal{L}(\mathrm{id}) = 0$ if X is horospherical.

Classification of rank one varieties

[Akhiezer 83], [Huckleberry-Snow 82]

Classification from:

- an explicit list of *cuspidal* cases (next slide), and
- a construction from these up to blowdown:

  given X G-spherical rank 1,
  there exists $\tilde{X} \to X$ birational, G-equivariant,

  such that

  $\tilde{X} \to G/P$ G-homogeneous fiber bundle over a rational homogeneous space G/P, with fiber a cuspidal rank 1 spherical S-variety, where S Levi subgroup of P.

For polarized manifolds, rank one spherical manifolds coincide with *cohomogeneity one manifolds*: manifolds equipped with a compact Lie group action with real hypersurface orbits. These have been instrumental in the development of canonical Kähler metrics (Calabi's extremal metrics on Hirzebruch surfaces leading to Calabi's ansatz, Koiso-Sakane first examples of non-homogeneous Fano Kähler-Einstein manifolds), but mostly considered when the cuspidal case is the toric $\mathbb{P}^1$ and there are no blowdowns. In other words, mostly considered homogeneous $\mathbb{P}^1$-bundles over generalized flag manifolds.

List of cuspidal cases from [Timashev]

G/H

Table 5.10: Wonderful varieties of rank 1

No.	G	H	$H \hookrightarrow G$	$\Pi^{\min}_{G/H}$	Wonderful embedding
1	$\mathrm{SL}_2 \times \mathrm{SL}_2$	SL_2	diagonal	$\omega + \omega'$	$X = \{(x:t) \mid \det x = t^2\} \subset \mathbb{P}(\mathrm{L}_2 \oplus \Bbbk)$
2	$\mathrm{PSL}_2 \times \mathrm{PSL}_2$	PSL_2		$2\omega + 2\omega'$	$\mathbb{P}(\mathrm{L}_2)$
3	SL_n	GL_{n-1}	symmetric No. 1 	$\omega_1 + \omega_{n-1}$	$\mathbb{P}^n \times (\mathbb{P}^n)^*$
4	PSL_2	PO_2	symmetric No. 3	$4\omega_1$	$\mathbb{P}(\mathfrak{sl}_2)$
5	Sp_{2n}	$\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$	symmetric No. 4	ω_2	$\mathrm{Gr}_2(\Bbbk^{2n})$
6	Sp_{2n}	$B(\mathrm{Sp}_2) \times \mathrm{Sp}_{2n-2}$		ω_2	$\mathrm{Fl}_{1,2}(\Bbbk^{2n})$
7	SO_n	SO_{n-1}	symmetric No. 6	ω_1	$X = \{(x:t) \mid (x,x) = t^2\} \subset \mathbb{P}^n$
8	SO_n	$\mathrm{S}(\mathrm{O}_1 \times \mathrm{O}_{n-1})$		$2\omega_1$	$\mathbb{P}^{n-1}$
9	SO_{2n+1}	$\mathrm{GL}_n \rightthreetimes \bigwedge^2 \Bbbk^n$		ω_1	$X = \{(V_1, V_2) \mid V_1 \subset V_1^\perp\} \subset \mathrm{Fl}_{n,2n}(\Bbbk^{2n+1})$
10	Spin_7	$\mathbf{G}_2$	non-symmetric No. 10	ω_3	$X = \{(x:t) \mid (x,x) = t^2\} \subset \mathbb{P}(V(\omega_3) \oplus \Bbbk)$
11	SO_7	$\mathbf{G}_2$		$2\omega_3$	$\mathbb{P}(V(\omega_3))$
12	$\mathbf{F}_4$	$\mathbf{B}_4$	symmetric No. 17	ω_1	
13	$\mathbf{G}_2$	SL_3	non-symmetric No. 12	ω_1	$X = \{(x:t) \mid (x,x) = t^2\} \subset \mathbb{P}(V(\omega_1) \oplus \Bbbk)$
14	$\mathbf{G}_2$	$N(\mathrm{SL}_3)$		$2\omega_1$	$\mathbb{P}(V(\omega_1))$
15	$\mathbf{G}_2$	$\mathrm{GL}_2 \rightthreetimes (\Bbbk \oplus \Bbbk^2) \otimes \bigwedge^2 \Bbbk^2$		$\omega_2 - \omega_1$	

Test configurations for spherical varieties

$M \otimes \mathbb{R}$

$\cup$

(X, L) polarized G-spherical variety, s B-section of L, $\mathrm{div}(s) = \sum_D n_D D$ and Δ associated polytope (defined by the equations $\rho(D)(m) + n_D \geq 0$)

Theorem [D]

- G-equivariant test configurations of (X, L) are in 1:1 correspondence with negative rational piecewise linear convex functions on the moment polytope Δ, whose slopes are in the opposite valuation cone $-\mathcal{V}$ of X.
- special test configurations correspond to linear functions $f \in -\mathcal{V}$
- product test configurations correspond to linear functions $l \in \mathrm{Lin}(\mathcal{V})$
- twists correspond to adding a linear fucntion $l \in \mathrm{Lin}(\mathcal{V})$.

$= \mathcal{V} \cap -\mathcal{V}$

First key remark: under the action of $G \times \mathbb{C}^*$, the total space $\mathcal{X}$ is still spherical
Why is it key? First for Odaka's remark: spherical varieties are Mori Dream Spaces. The notion of K-stability used by Chi Li reduces to G-uniform K-stability thanks to this property.

Convex function associated to a test configuration

$\mathcal{X}_0$ is a $G \times \mathbb{C}^*$-stable divisor in the spherical variety $\mathcal{X}$
Recall that s is a fixed B-equivariant holomorphic section of L. Extend s^r to a $B \times \mathbb{C}^*$-equivariant section of $\mathcal{L}$ using the isomorphism $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L^r)$ and the $\mathbb{C}^*$-action.
Associated divisor has to be of the form

reduced
$\mathcal{X}_0 = \sum D_j$

$$r\overline{\mathrm{div}(s) \times \mathbb{C}^*} + \sum_j n_j D_j$$

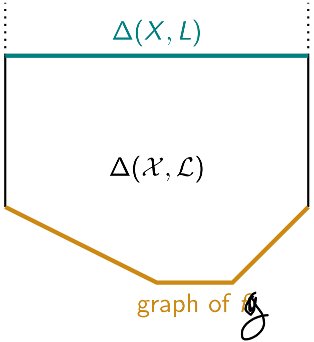
D_j prime $G \times \mathbb{C}^*$-stable divisors

for some $G \times \mathbb{C}^*$-stable prime divisors D_j, corresponding to some primitive elements $(u_j, t_j) \in N \times \mathbb{Z}_{<0}$.
The PL function is

$\Delta \longrightarrow \mathbb{R}$

$$g : x \mapsto \sup \frac{r u_j(x) + n_j}{t_j}$$

Picture



Special test configurations

"Satellites"

It follows from our description of special test configurations + [Batyrev-Moreau] that there are only a finite number of possible central fibers of special test configurations: the central fiber depends only on the face of $-\mathcal{V}$ to which the associated linear function belongs.

Note there are still infinitely many special test configurations: the $\mathbb{C}^*$ action can change. But they form a "finite dimensional" space.

Consequence: [Li-Han]'s G-uniform K-stability with respect to special test configurations is equivalent to G-equivariant stability with respect to special test configurations!

In my first approach to K-stability, I used the horospherical degeneration: if the linear function belongs to the interior of $-\mathcal{V}$, then the central fiber is horospherical, coincides with Popov's general horospherical degeneration. All central fibers of special test configurations for X, L admit a further test configuration with central fiber the horospherical degeneration of the original X, L.

NA functionals: first step

$$\frac{\sum \lambda_{j,k}}{k d_k} = F_0 - \frac{DF}{k} + o\left(\frac{1}{k}\right)$$

Donaldson's observation: Using restriction exact sequences, get

$$\sum_{j,k} \lambda_{j,k} = \dim H^0(\mathcal{X}, \mathcal{L}^k) - \underbrace{\dim H^0(\mathcal{X}_0, \mathcal{L}_0^k)}_{\dim H^0(X, L^{rk})}$$

Toric case: all these dimensions may be computed by counting integral points in a convex polytope.
Spherical case: add values of a polynomial (using Weyl dimension formula) at integral points in a convex polytope

Khovanski-Pukhlikov's theorem

Basic toric case:

$$\lim_{k\to\infty} \frac{\# k\Delta \cap \mathbb{Z}^n}{k^n} = \mathrm{Vol}(\Delta)$$

where volume is wrt Lebesgue measure normalized by lattice $\mathbb{Z}^n$.
Actually, expansion, (Riemann-Roch formula for polytopes)

$$\begin{aligned}\dim H^0(X, L^k) &= \sum_{\lambda \in M \cap k\Delta} \dim V_{k\chi+\lambda} \\ &= k^n \int_\Delta P d\mu + k^{n-1}\left(\frac{1}{2}\int_{\partial\Delta} P d\sigma + \int_\Delta Q d\mu\right) + o(k^{n-1})\end{aligned}$$

where $d\sigma$ restricted to any facet is the Lebesgue measure normalized by the intersection of M with the linear space generated by the facet.

$$\dim\ V_{k\lambda} = k^d P(\lambda) + k^{d-1} Q(\lambda) + \cdots \text{ given by Weyl dimension formula}$$

special case of [Pukhlikov-Khovanskii, *The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes*, 1992]

NA functionals: conclusion

Set

$$P(x) = \prod_{\alpha \in \Phi_X^+} \frac{\langle x+\chi, \alpha\rangle}{\langle \varpi, \alpha\rangle} \qquad Q(x) = \sum_{\alpha \in \Phi_X^+} \frac{\langle \varpi, \alpha\rangle}{\langle x+\chi, \alpha\rangle} P(x)$$

$$V = \int_\Delta P d\mu \qquad a = \frac{1}{2V}\left(\int_{\partial\Delta} P d\sigma - \int_\Delta Q d\mu\right)$$

Theorem [D]

Let f be the convex PL function associated to $(\mathcal{X}, \mathcal{L})$.

$$M^{NA}(\mathcal{X}, \mathcal{L}) = \frac{1}{2V}\left(\int_{\partial\Delta} fP d\sigma + \int_\Delta f2(Q - aP) d\mu\right) =: \frac{1}{2V}\mathcal{L}(f)$$

and

$$J^{NA}(\mathcal{X}, \mathcal{L}) = \frac{1}{V}\int_\Delta (f - \min f) P d\mu =: \frac{1}{V}\mathcal{J}(f)$$

Uniform K-stability of polarized spherical varieties

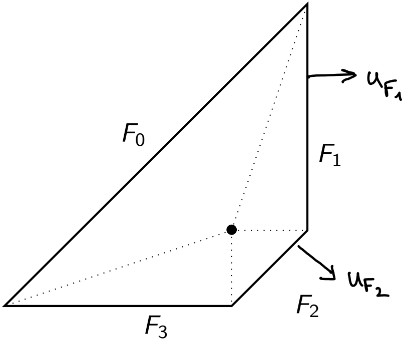
$$\mathcal{L}(f) = \int_{\partial\Delta} fPd\sigma + \int_{\Delta} f2(Q - aP)d\mu$$

$$\mathcal{J}(f) = \int_{\Delta} (f - \min f)Pd\mu$$

(X, L) is G-uniformly K-stable if and only if there exists $\epsilon > 0$ such that for all convex PL function f on Δ with slopes in $-\mathcal{V}$,

$$\mathcal{L}(f) \geq \epsilon \inf_{l \in \mathrm{Lin}(\mathcal{V})} \mathcal{J}(f + l)$$

Towards a combinatorial sufficient condition



Choose a point 0 in the interior of Δ, decompose Δ into pyramids T_F with base the facet F and vertex 0, as the facets F vary.
Let u_F denote the primitive outward normal to the facets of Δ, and let n_F be the numbers such that

$$\Delta = \{x \mid \underbrace{u_F(x) \leq n_F}\}$$

A sufficient condition

Let (X, L) be a polarized G-spherical variety

Theorem

Assume that for all F and $x \in T_F$,

$$d_x P(x) + (r+1)P(x) + 2n_F(Q - aP)(x) \geq 0$$

then (X, L) is G-uniformly K-stable if and only if (X, L) is K-stable with respect to special test configurations.

Corollary

Assume that X is smooth. Assume that for all F and $x \in T_F$,

$$d_x P(x) + (r+1)P(x) + 2n_F(Q - aP)(x) \geq 0$$

then there exists a cscK metric in $c_1(L)$ if and only if (X, L) is K-stable with respect to special test configurations.

Sketch of proof

1. By approximation, can work with smooth convex functions instead of PL convex functions.
2. Normalize functions: work with f smooth convex on Δ, such that $\inf f = 0$ and $d_0 f$ is in a fixed complement subspace $\mathcal{W}$ to $\mathrm{Lin}(\mathcal{V})$. Can always reduce to this by adding an affine function with slope in $\mathrm{Lin}(\mathcal{V})$.
3. On normalized functions, G-uniform K-stability writes $\mathcal{L}(f) \geq \epsilon \int_\Delta fPd\mu$
4. Use divergence formula to transform the integral on the boundary to integrals on the interior of the polytope:

$\int_{\partial\Delta} gPd\sigma$

$$\int_F fPd\sigma = \frac{1}{n_F}\int_{T_F} \left(P(x)d_xf(x) + rf(x)P(x) + f(x)d_xP(x)\right)d\mu$$

$$\mathcal{L}(f) = \int_{\partial\Delta} fPd\sigma - \int_\Delta f(2(aP-Q))d\mu$$

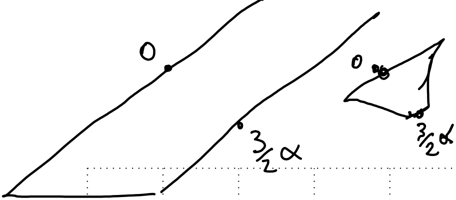
5. Plug into $\mathcal{L}$

$$\mathcal{L}(f) = \sum_F \frac{1}{n_F}\int_{T_F} \left(d_xf(x) - f(x)\right)P(x)d\mu$$
$$+ \sum_F \frac{1}{n_F}\int_{T_F} \left((r+1)P(x) + d_xP(x) - 2n_F(aP-Q)(x)\right)f(x)d\mu$$

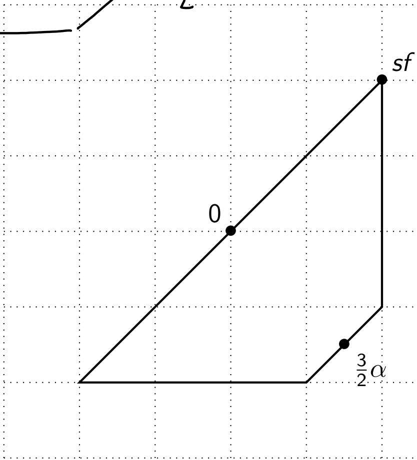
K-semistability immediate.

uniform K-stability involve a compactness argument

An example



Consider the $\mathrm{SL}_2 \times \mathbb{C}^*$-spherical variety $\mathrm{Bl}_{Q^1} Q^3$. Let α be the unique positive root and f the generating character of $\mathbb{C}^*$. Up to scaling, the moment polytope of an ample line bundle is as on the right



By the sufficient condition, the associated Kähler class admits a cscK metric if $1,683 < s < 3$

Case of Fano toric manifolds

Inspiration: a sufficient condition for properness of (modified) Mabuchi functional on toric manifolds [Zhou-Zhu 2008]

Recall condition: for $x \in T_F$,

$$d_x P(x) + (r+1)P(x) + 2n_F(Q - aP)(x) \geq 0$$

In toric case, $P = 1, Q = 0$, get $\forall F$ $r = \dim(X)$

$$r + 1 - 2n_F a \geq 0$$

Notable particular case: all n_F are the same, then the condition is that a given polynomial is non-negative on the whole polytope, a particularly nice condition.
All $n_F = 1$ means the polytope Δ is *reflexive*, which is equivalent to the associated toric manifold being Fano $r+1-2a \geqslant 0$
Furthermore, $2a =$ scalar curvature, which in Fano case, for L the anticanonical line bundle, is $= r$ the dimension of the toric manifold
Thus $r + 1 - r = 1 > 0$
The condition varies continuously $\Rightarrow$ condition holds on a neighborhood of $c_1(X)$.

Case of close to Fano spherical manifolds

In the general spherical case, it turns out that the condition is also always satisfied for the anticanonical class.

It is more difficult to check: the moment polytope is no longer reflexive.

But, there is a precise description [Brion, Pasquier, Gagliardi-Hofscheier] which allows to deal with all cases. In particular, when a facet does not have $n_F = 1$, then P actually vanishes on that facet, and the condition associated to that facet may be written differently (can work as if $n_F = 1$).
Since linear functions are smooth, can use the new expression of $\mathcal{L}$ in the proof of sufficient criterion to recover

Theorem

Combinatorial criterion for KE metrics on spherical manifolds from Lecture 1.

Conjecture

On a neighborhood of $c_1(X)$, uniform K-stability is equivalent to K-stability with respect to special test configurations.

[D]: proof for large classes of spherical varieties (open orbit affine with trivial Picard group, toroidal horospherical), but not all.

Back to rank 1 $\mathcal{L}(g) = \int_{\partial\Delta} gP\,d\sigma - \int_\Delta g\,2(aP-Q)\,d\mu$

$\Delta = [S_-, S_+] \subset \mathbb{R}$

g convex normalized: $\inf g = 0$

+ $g(0)=0$ if horosph | g increasing if not

WTS: K-stability wrt special t.c. $\Rightarrow$ unif K-stab

ie. $\mathcal{L}(g) \geq 0$ on affine
$= 0$ iff $g \in Lin(\nu)$ $\Longrightarrow$ $\mathcal{L}(g) \geq \varepsilon \int_\Delta gP\,d\mu$

Look at minimizer of $\dfrac{\mathcal{L}(g)}{\int_\Delta gP\,d\mu}$

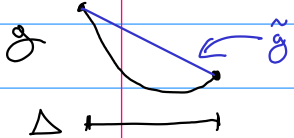
Case of $\mathbb{P}^1$:

$$\mathcal{L}(g) = \underbrace{\int_{\partial\Delta} g\, d\sigma}_{\text{positive contrib}} - \underbrace{\int_{\Delta} 2a\, g\, d\mu}_{\text{negative contrib}}$$

minimize $\mathcal{L}(g)$: g as small as possible on $\partial\Delta$

as large as ——— $\mathrm{Int}(\Delta)$

by compactness, $\exists g$ that minimizes $\mathcal{L}$



$\mathcal{L}(\tilde{g}) \leq \mathcal{L}(g)$

& $\tilde{g}$ is affine!

remains to show that $\tilde{g} \notin \mathrm{Lin}(\nu)$.

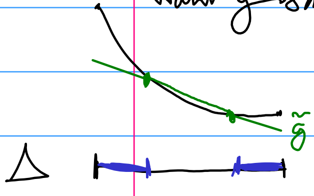
general case:

$$\mathcal{L}(g) = \int_{\partial\Delta} gP\,d\sigma - \int_{\Delta} g\,2(aP-Q)\,d\mu$$

$aP-Q$ changes sign

positive contrib: $\displaystyle\int_{\partial\Delta} gP\,d\sigma - \int_{\{aP-Q<0\}} g\,2(aP-Q)\,d\mu$

want g small on $\partial\Delta \cup \{aP-Q<0\}$



$\tilde{g}$ minimizer affine.

Alternative approach

Use an analytical, variational approach:

- compute the Mabuchi functional,
- use Chen-Cheng directly.

Would imitate the proof [Donaldson]+[Zhou-Zhu]+[Chen-Cheng]

At least two obstacles, but would be of interest to tackle these issues:

1. deal with the differential geometry of more general spherical varieties than horosymmetric varieties
2. even in horosymmetric case, some obstacles to be lifted (e.g. Hermitian case, but not only cf my Crelle paper)