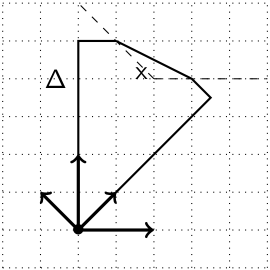
Existence of canonical Kähler metrics on spherical varieties — Lecture 1

Ensemble of Algebra and Geometry



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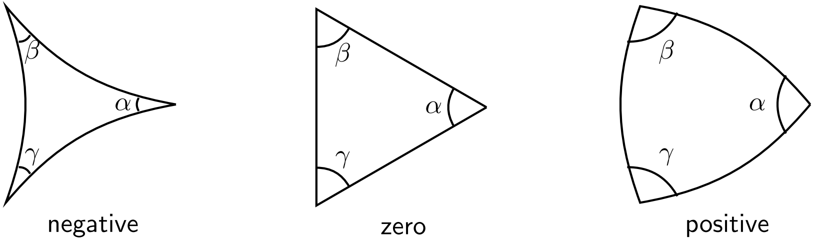
Motivation

Riemann uniformization theorem :

Every real oriented compact surface
admits
a constant curvature Riemannian metric

Curvature :



Kähler metrics

X compact complex manifold
Kähler metric g on X $\Leftrightarrow$ Kähler form ω on X

Local definition

A global 2-form ω on X is a Kähler form if it writes in local holomorphic coordinates $(z_1, \ldots z_n)$ as

$$\begin{aligned}\omega &= i\,\partial\bar{\partial}\,\phi \\ &:= i \sum_{j,k} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k\end{aligned}$$

where ϕ real valued local smooth function and $\left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}\right)$ is a positive definite Hermitian matrix everywhere.

It is a closed, real, positive $(1,1)$-form on X. It defines a de Rham cohomology class $[\omega]$ called a Kähler class. A complex manifold X is called Kähler if it admits a Kähler class.

Baby example: the projective line

The complex projective line $\mathbb{P}^1$

$$\begin{aligned}\mathbb{P}^1 &= \mathbb{C}^2 \setminus \{0\}/\mathbb{C}^* = \{[x:y] \mid (x,y) \in \mathbb{C}^2 \setminus \{0\}\} \\ &= \mathrm{GL}_2 \,/ \begin{pmatrix} \mathbb{C}^* & \mathbb{C} \\ 0 & \mathbb{C}^* \end{pmatrix} = \mathrm{SU}(2)/S(U(1) \times U(1))\end{aligned}$$

As a complex manifold, covered by two coordinate charts

$$\mathbb{C} \to \mathbb{P}^1, x \mapsto [x:1] \qquad \text{and} \qquad \mathbb{C} \to \mathbb{P}^1, y \mapsto [1:y] \qquad \text{glued by} \qquad x \mapsto \frac{1}{x}$$

is equipped with Fubini-Study Kähler form:

$$\omega_{FS} = i\,\partial\bar{\partial}\ln(1+|x|^2) = \frac{idx \wedge d\bar{x}}{(1+x\bar{x})^2}$$

More generally, ω_{FS} Fubini-Study metric on

$$\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^* = \mathrm{SU}(n+1)/S(U(1) \times U(n))$$

Curvature forms from Hermitian line bundles

Let L be a holomorphic line bundle on X

Curvature of a Hermitian metric

Let h be a Hermitian metric on L (Hermitian norm on each fiber, varying smoothly). Define its curvature (a global closed real $(1,1)$-)form ω_h locally by: if s local frame (trivializing holomorphic section) of L,

$$\omega_h = i\,\partial\bar{\partial}(-\ln|s|_h^2)$$

Does not depend on choice of s: if f nowhere-zero holomorphic function, $\partial\bar{\partial}\ln|f| = 0$. Note also that a multiple of h has the same curvature ω_h.

Example

On $(\mathbb{P}^n, O(1))$, the unique $\mathrm{SU}(n+1)$-invariant Hermitian metric on $O(1)$ (up to multiple) has curvature ω_{FS} the Fubini-Study metric.

Ampleness and Kähler forms

The (de Rham) cohomology class defined by the closed 2-form ω_h depends only on L, it is denoted by $c_1(L)$. recall

$\partial\bar{\partial}$-Lemma

ω_1 and ω_2 are in the same cohomology class if $\omega_1 - \omega_2 = i\,\partial\bar{\partial}\,\psi$ for some $\psi : X \to \mathbb{R}$)

Say h is positively curved if ω_h is Kähler.

Theorem [Kodaira]

L is ample iff there exists a positively curved Hermitian metric h on L

if L is very ample, get Kodaira embedding $X \to \mathbb{P}(H^0(X, L)^* \simeq \mathbb{P}^N$ such that L coincides with restriction of $O(1)$. Restriction of above metric provides a positively curved metric on L. Its curvature is the restriction of the Fubini-Study metric on $\mathbb{P}^N$.

Kähler-Einstein metrics

There are various measures of curvature, each yielding possible definitions of canonical Kähler metrics. For now,

Ricci curvature form

Given ω Kähler, it is the global closed real $(1,1)$-form defined locally by

$$\mathrm{Ric}(\omega) = i\,\partial\bar{\partial}\left(-\ln\det\frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}\right)$$

e.g. on $\mathbb{P}^1$, $\mathrm{Ric}(\omega_{FS}) = i\,\partial\bar{\partial}\left(-\ln\frac{1}{(1+x\bar{x})^2}\right) = 2\omega_{FS}$

Kähler-Einstein metric

A Kähler form ω is Kähler-Einstein if it satisfies the Kähler-Einstein equation:

$$\mathrm{Ric}(\omega) = t\omega \qquad\qquad (KE)$$

for some real number t

Uniformization theorem $\Rightarrow$ all Riemann surfaces admit KE metrics
e.g. ω_{FS} on $\mathbb{P}^1$ for $t = 2$

First Chern class

If ω arbitrary Kähler form on X, then ω defines a Hermitian metric on the canonical bundle K_X^{-1}. Recall $K_X = \det \Omega_X$ where Ω_X holomorphic cotangent bundle. That is, K_X is the bundle of holomorphic volume forms. For $0 \neq \xi \in K_X^{-1}$ with dual $0 \neq \xi^* \in K_X$, set

$$|\xi|_h^2 := \frac{|\xi^* \wedge \overline{\xi^*}|}{\omega^n/n!}$$

The Ricci curvature form $\mathrm{Ric}(\omega)$ **is the curvature of this Hermitian metric:** in local holomorphic coordinates $(z_1, \ldots, z_n)$, take $\xi^* = dz_1 \wedge \cdots \wedge dz_n$ and note that locally

$$\frac{\omega^n}{n!} = \det\left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}\right) |\xi^* \wedge \overline{\xi^*}|$$

The corresponding class is the first Chern class of X, denoted by $c_1(X)$.

First obstruction to KE metrics

In dimension higher than 1, many manifolds do not admit KE metrics.
At the cohomology class level,

$$\mathrm{Ric}(\omega) = t\omega \quad \Longrightarrow \quad c_1(X) = t[\omega]$$

in particular, if $t \neq 0$, $\frac{1}{t} c_1(X)$ is a Kähler class.

Three cases:

1. $c_1(X) < 0$ e.g. hyperbolic Riemann surface
2. $c_1(X) = 0$ e.g. compact complex torus
3. $c_1(X) > 0$ e.g. $\mathbb{P}^1$

Many manifolds do not have definite or zero $c_1(X)$ e.g. product of above.

Aubin-Calabi-Yau theorem

Calabi problem

Existence? Uniqueness of KE metric?

(partial but revolutionary) answer:

Calabi-Yau theorem [Aubin-Yau]

1. $c_1(X) < 0$ there always exists a unique KE $\omega \in \frac{1}{t} c_1(X)$ [Aubin-Yau]
2. $c_1(X) = 0$ for every Kähler class α, there exists a unique KE $\omega \in \alpha$ [Yau]

In the last case $c_1(X) > 0$ (equivalently, K_X^{-1} ample), X is called a Fano manifold, and there does not always exist a KE metric! The existence problem is very subtle.

Uniqueness [Bando-Mabuchi]

if ω_1 and ω_2 are two KE metrics, there exists $g \in \mathrm{Aut}(X)$ such that $\omega_2 = g^* \omega_1$.

When $\mathrm{Aut}(X)$ is positive dimensional, get infinitely many KE metrics, but they form an orbit of $Aut(X)$ isomorphic to the symmetric space $\mathrm{Aut}(X)/K$.

Now a positive result

We have shown that $\mathbb{P}^1$ is KE.
More generally:

Proposition

If a compact Lie group acts transitively by biholomorphisms on a Fano manifold X, then X admits a KE metric.

This is not completely obvious: there are many K-invariant Kähler metrics on X, not all are KE metrics.
But: there is up to obvious constants a unique K-invariant Hermitian metric on K_X^{-1}! Take its curvature, it is KE.
These manifolds bear different names: rational homogeneous spaces, generalized flag manifolds,...

Examples

The projective space $\mathbb{P}^n$, Grassmannians $\mathrm{Grass}(k, n)$, quadrics Q^n,...

Matsushima's obstruction

Theorem [Matsushima 1957]

Assume that X is a Fano manifold that admits a Kähler-Einstein metric ω, and let $K := \mathrm{Isom}(\omega)$ isometry group. Then

1. $\mathrm{Aut}(X)$ is a complex reductive group
2. K is a maximal compact subgroup of $\mathrm{Aut}(X)$

More precisely if ω is Kähler-Einstein, , then $\mathrm{Aut}(X) = K^{\mathbb{C}}$.
Reductive group: $G = K^{\mathbb{C}}$ for some compact real Lie group may be taken as definition: $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ and $G = K \exp(i\mathfrak{k})$.
Or recall from Brion's lecture definition in terms of radical.

For linear reductive group, it is equivalent to G being the quotient of a product of simple complex Lie groups and of a tori $(\mathbb{C}^*)^k$ by a finite central subgroup.

Examples

Simple complex Lie groups: $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{PSO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$
Reductive not semisimple: $\mathrm{GL}_n = \mathrm{U}(n)^{\mathbb{C}} = \frac{\mathrm{SL}_n \times \mathbb{C}^*}{\mu_n}$

An example

G^0 denotes the maximal connected subgroup of a topological group G.
Note that G is reductive if and only if G^0 is.

Blanchard's Lemma

Let $f : X \to Y$ be a proper morphism with $f_*\mathcal{O}_X = \mathcal{O}_Y$, then there exists a unique action of $\mathrm{Aut}^0(X)$ on Y such that f is $\mathrm{Aut}^0(X)$-equivariant.

For the blowup $X = \mathrm{Bl}_Z Y \to Y$ of Y at submanifold Z, get $\mathrm{Aut}^0(X) \subset \mathrm{Stab}_{\mathrm{Aut}(Y)} Z$ and reverse monomorphism by universal property of blowup, hence an isomorphim.

A Fano manifold with non-reductive automorphism group

$\mathrm{Aut}^0(\mathrm{Bl}_{\mathbb{P}^k}\mathbb{P}^n) = \mathbb{P}\begin{pmatrix} \mathrm{GL}_{k+1} & \mathrm{M}_{k+1,n-k} \\ 0 & \mathrm{GL}_{n-k} \end{pmatrix}$ non-reductive

e.g. simplest case $k = 0$, $n = 2$, $\dim_{\mathbb{C}} \mathrm{Aut}^0(X) = 6$, but maximal compact subgroup is $\mathbb{P}\begin{pmatrix} U(1) & 0 \\ 0 & U(2) \end{pmatrix}$ with real dimension 4

Del Pezzo surfaces: Tian's Theorem

Fano manifolds of dimension 2 are called Del Pezzo surfaces.

There are only a few deformation classes:

1. $\mathbb{P}^1 \times \mathbb{P}^1$
2. the various blowups of $\mathbb{P}^2$ at up to 8 points.

We have already seen that $\mathbb{P}^1 \times \mathbb{P}^1$ (product metric) and $\mathbb{P}^2$ are KE, and that $\mathrm{Bl}_{1\ \mathrm{pt}}\mathbb{P}^2$ is not KE.
Full answer is known and the only obstruction is Matsushima's.

Theorem [Tian]

A Del Pezzo surface admits a KE metric if and only if its automorphism group is reductive.

In other words, all Del Pezzo surfaces but the blowup of $\mathbb{P}^2$ at one or two points are KE, since $\mathrm{Aut}^0(\mathrm{Bl}_{2\ \mathrm{pts}}\mathbb{P}^2) = \mathbb{P}\begin{pmatrix} \mathbb{C}^* & 0 & \mathbb{C} \\ 0 & \mathbb{C}^* & \mathbb{C} \\ 0 & 0 & \mathbb{C}^* \end{pmatrix}$, $\mathrm{Aut}^0(\mathrm{Bl}_{3\ \mathrm{pts}}\mathbb{P}^2) = (\mathbb{C}^*)^2$ and $\mathrm{Aut}^0(\mathrm{Bl}_{\geq 4\ \mathrm{pts}}\mathbb{P}^2) = \{1\}$

Futaki's obstruction

X Fano manifold. Note that up to scaling, can search for KE metrics in $c_1(X)$.
Let $\omega \in c_1(X)$. Since ω and $\mathrm{Ric}\,\omega$ are in the same class, can write

$$\mathrm{Ric}(\omega) - \omega = i\,\partial\bar{\partial}\,h$$

Then $\quad \omega$ KE $\quad$ iff $\quad \partial\bar{\partial}\,h = 0 \quad$ iff $\quad h$ is constant.
Let ξ be a holomorphic vector field on X, which may be identified with an element of $\mathfrak{aut}(X)$ the Lie algebra of $\mathrm{Aut}(X)$.

Theorem [Futaki]

The following is independent of the choice of $\omega \in c_1(X)$:

$$\mathrm{Fut}(\xi) := \int_X (\xi \cdot h)\omega^n$$

Furthermore, $\mathrm{Fut} : \mathfrak{aut}(X) \to \mathbb{R}$ defines a Lie algebra character.

Corollary

$\mathrm{Fut} \neq 0$ implies that X does not admit KE metrics.

Examples

One can check that the Futaki invariant of $\mathrm{Bl}_{\mathbb{P}^k}\mathbb{P}^n$ is non-zero. But:

The two obstructions are different:

1. there are Fano manifolds with non-reductive automorphism group and vanishing Futaki invariant (e.g. there exists Fano threefolds with automorphism group the additive group $\mathbb{C}$ and vanishing Futaki character).
2. Futaki's example $\mathrm{Bl}_{\mathbb{P}^1,\mathbb{P}^2}\mathbb{P}^4$ has reductive automorphism group but non-zero Futaki invariant.

More generally: $\mathrm{Bl}_{\mathbb{P}^k,\mathbb{P}^{n-k-1}}\mathbb{P}^n$ has vanishing Futaki invariant if and only if $n = 2k+1$, but

$$\mathrm{Aut}^0(\mathrm{Bl}_{\mathbb{P}^k,\mathbb{P}^{n-k-1}}\mathbb{P}^n) = \mathbb{P}\begin{pmatrix} \mathrm{GL}_{k+1} & 0 \\ 0 & \mathrm{GL}_{n-k} \end{pmatrix} = \mathbb{P}\begin{pmatrix} \mathrm{U}(k+1) & 0 \\ 0 & \mathrm{U}(n-k) \end{pmatrix}^{\mathbb{C}}$$

We will see how to compute Futaki invariant of many examples in the next lectures!

Tian's example and greatest Ricci lower bound

It was originally hoped that Futaki's obstruction was a necessary and sufficient condition. Tian proved that it is not the case, while initiating the study of K-stability.

Mukai-Umemura deformations

There exists a Fano theefold, called the (a?) Mukai-Umemura threefold, which admits a KE metric, some of whose deformations are not KE.

These manifolds actually admit Kähler metrics that are arbitrarily close to being KE:

Greatest Ricci lower bound

$$GRLB(X) := \sup\{t \in [0,1] \mid \exists\, \omega \in c_1(X), \quad \mathrm{Ric}(\omega) \geq t\omega\}$$

It is an invariant of a Fano manifold X measuring how far from being KE it is:

$$X \text{ KE} \implies GRLB(X) = 1$$

K-stability and the YTD conjecture

The previous example lead to the Yau-Tian-Donaldson (YTD) conjecture:

there exists a KE metric on X iff X is K-stable

The initial idea was to consider the Futaki invariant, not of the variety of interest X, but of some of its degenerations. Ding and Tian showed first that under certain conditions, this Futaki invariant must take positive values.

Theorem [Chen-Donaldson-Sun and Tian, 2015]

The YTD conjecture is true.

There are variants of the YTD conjecture for other types of canonical Kähler metrics, still open today.

I will say more on K-stability and the YTD conjecture on Friday during Lecture 3.

What to do now?

Given a *known* manifold, how to check effectively if it is KE?

Several directions:

1. delta invariant, valuative approach and moduli approach
2. Manifolds with large group actions

Lecture 1 today: what kind of results can we hope for in the second direction.

If there are no KE metrics, what alternative canonical Kähler metrics?

1. coupled generalized solitons
   Lecture 2: differential geometric approach to these
2. cscK, extremal Kähler metrics
   Lecture 3: algebro geometric approach to these

Recollections on reductive groups: root system

We shall now focus on spherical varieties.
First, let's recall some key features of the theory of reductive groups.

Let G be a connected complex reductive group, B a Borel subgroup of G and $T \simeq (\mathbb{C}^*)^N$ a maximal torus of B

$\mathrm{X}^*(T) := \{\chi : T \to \mathbb{C}^* \text{ morphism}\} \simeq \mathbb{Z}^N$ group of characters of T.

$\Phi \subset \mathrm{X}^*(T)$ root system of (G, T), $\Phi^+ \subset \Phi$ roots of B.

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \qquad \mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall t \in T, \mathrm{Ad}(t)(x) = \alpha(t)x\}$$

Example: GL_n, B upper triangular matrices, T diagonal matrices

$\mathrm{X}^*(T)$ generated by $\mathrm{diag}(a_1, \dots, a_n) \mapsto a_j$
Φ is the set of $\alpha_{j,k} : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_j/a_k$ for $j \neq k$, and $\mathfrak{g}_{\alpha_{j,k}} = \mathbb{C}E_{j,k}$
$\alpha_{j,k} \in \Phi^+$ iff $j < k$.

Recollections on reductive groups: representations

(as always, working over $\mathbb{C}$)
We fix $\langle\cdot,\cdot\rangle$ a scalar product on $\mathrm{X}^*(T)\otimes\mathbb{R}$ extending the Killing product.
(can see $\mathrm{X}^*(T)$ inside $\mathfrak{g}$, such that $\mathrm{X}^*(T)\otimes\mathbb{R}\simeq i\mathfrak{k}\cap\mathfrak{t}$).

1. All finite dimensional representations of G are decomposable into direct sums of irreducible representations.
2. There is a bijection between the set of dominant weights $\{\chi\in\mathrm{X}^*(T)\mid\forall\alpha\in\Phi^+,\langle\alpha,\chi\rangle\geq 0\}$ and the set of irreducible representations of G up to isomorphism.
3. Explicitely, sending an irreducible representation V to the weight χ of the unique B-eigenvector in V, called the highest weight of V.

We denote by V_χ an irreducible representation with highest weight χ.

$(\Phi^+)^\vee:=\{\chi\in\mathrm{X}^*(T)\otimes\mathbb{R}\mid\forall\alpha\in\Phi^+,\langle\alpha,\chi\rangle\geq 0\}$ called the positive Weyl chamber

Recollection on G-varieties: Moment polytope

(X, L) polarized G-variety (equipped with an action of a connected complex reductive group G as before, the action on L being linearized)

Moment polytope

$$\Delta = \Delta(X, L) = \mathrm{Conv}\left\{\frac{\lambda}{k}\right\}$$

where $k \in \mathbb{Z}_{>0}$ and λ runs over all characters of B such that there exists a B-eigensection $s \in H^0(X, L^k)$ with eigenvalue λ:

$$\forall b \in B, \quad b \cdot s = \lambda(b)s$$

This is a convex polytope sitting inside the positive Weyl chamber of (G, T, B).

Note:

- Δ depends on the G-linearization.
- G-linearizations of the same line bundle differ by a character of G
- If $L = K_X^{-1}$ then there is a canonical G-linearization, thus a canonical moment polytope.

Recollections on spherical manifolds 1

Definition

A normal G-variety X is spherical if B acts with an open (and dense) orbit on X.

Implies that G also has an open dense orbit G/H.
Call $H \subset G$ a spherical subgroup if G/H is a spherical variety.

Weight lattice

The weight lattice $M = M(X)$ of a G-spherical variety X is the set of all characters λ of B such that there exists a B-equivariant rational function f on X with weight λ:

$$\forall b \in B, \quad b \cdot f = \lambda(b) f$$

where $b \cdot f(x) = f(b^{-1} \cdot x)$.

Note:

- such a function is uniquely determined by its weight λ up to a constant
- $\mathrm{X}^*(B) = \mathrm{X}^*(T)$ so M lives in the same space as Φ
- Weight lattice depends only on open G-orbit G/H.

Recollections on spherical manifolds 2

A valuation of $\mathbb{C}(X)$ (the field of rational functions on X) is a group morphism $\nu : \mathbb{C}(X)^* \to \mathbb{R}$ such that $\nu(\mathbb{C}^*) = \{0\}$ and $\nu(f_1 + f_2) \geq \min \nu(f_i)$.

Let $N := \mathrm{Hom}(M, \mathbb{Z})$. The restriction of a valuation ν to B-semi-invariant rational functions produces an element $\rho(\nu) \in N \otimes \mathbb{R}$.

Valuation cone

The valuation cone $\mathcal{V}$ of X is the image by ρ of the set of G-invariant valuations of $\mathbb{C}(X)$. It is a rational polyedral cone in $N \otimes \mathbb{R}$.

Again, $\mathcal{V}$ depends only on the open orbit G/H
Actually, M, $\mathcal{V}$ + data of *color map* fully encode G/H:

Consider the (finite) set $\mathcal{C}(G/H)$ of B-stable prime divisors of G/H (irreducible components of the complement of the open B-orbit). Identify $\mathcal{C}(G/H)$ with a set of valuations of $\mathbb{C}(G/H)$ (to a function f, associate its order of vanishing along the divisor).
The *color map* is the restriction of ρ to $\mathcal{C}(G/H)$, seen as an abstract map from a finite set to $N \otimes \mathbb{R}$.

Moment polytope for spherical varieties

X spherical G-variety, L ample G-linearized line bundle on X.
Moment polytope Δ + weight lattice M fully encode the G-representation structure of $H^0(X, L)$: fix $s \in H^0(X, L^k)$ a B semi-invariant section with weight χ, then

$$H^0(X, L^k) = \bigoplus_{\lambda \in k\Delta; \lambda - \chi \in M} V_\lambda$$

where V_λ irreducible G representation with highest weight λ
In particular, multiplicities are zero or one for all dominant weights, which explains the other name *multiplicity free variety* (the notion is actually a bit different if one does not consider only polarized varieties).
In particular, get an expression for the dimension of $H^0(X, L^k)$, thanks to:

Weyl dimension formula

$\dim V_\lambda = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \varpi, \alpha \rangle}{\langle \varpi, \alpha \rangle}$ where $\varpi = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

KE metrics on Fano toric manifolds [Wang-Zhu]

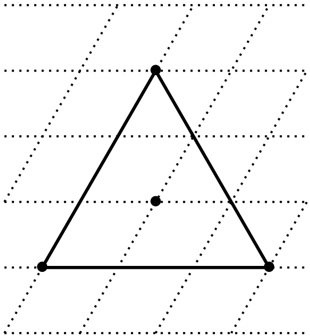
When $G = B = T$, a spherical manifold is a toric manifold. (slight difficulty: beware of the conventions for the fans which are not always the same + the action of T is not required to be effective).
Assume X is Fano and let $\Delta \subset N \otimes \mathbb{R}$ be its (canonical) moment polytope.

Theorem [Wang-Zhu, 2004]

X admits a KE metric if and only if $\mathrm{Bar}(\Delta) = 0$.

$\mathrm{Bar}(\Delta) = \frac{\int_\Delta p dp}{\int_\Delta dp}$ where dp Lebesgue measure

Examples:
Projective plane $\mathbb{P}^2$



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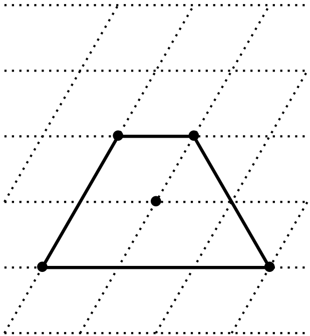
Examples:

Projective plane blown up at one point



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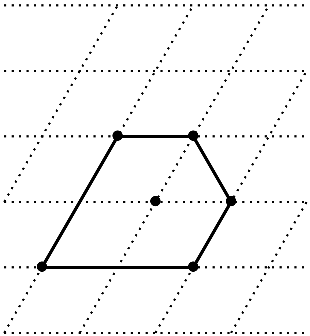
Examples:

Projective plane blown up at two point



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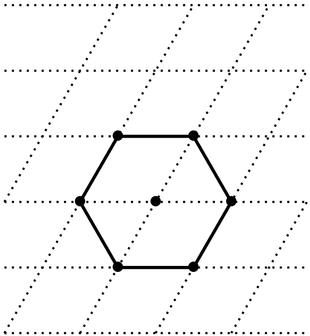
Examples:
Projective plane blown up at three point



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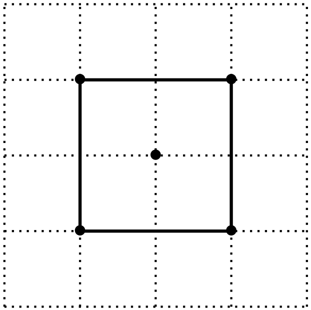
Examples:
Product of two projective lines
$\mathbb{P}^1 \times \mathbb{P}^1$



KE metrics for Fano spherical manifolds

Back to X spherical G-manifold, assume X Fano and Δ its anticanonical moment polytope.
Let $\Phi_X^+ := \{\alpha \in \Phi^+ \mid \exists p \in \Delta, \langle \alpha, p \rangle \neq 0\}$ and $\varpi_X := \frac{1}{2}\sum_{\alpha\in\Phi_X^+} \alpha$
Note that these data depend only on the open orbit G/H.

Theorem [D.2020]

X admits a KE metric if and only if the Duistermaat-Heckman barycenter translated by $-2\varpi_X$ is in the relative interior of the opposite of the cone dual to the valuation cone, in formulas:

$$\mathrm{Bar}(\Delta) - 2\varpi_X \in \mathrm{Relint}(-\mathcal{V}^\vee)$$

where

$$\mathrm{Bar}(\Delta) = \frac{\int_\Delta p \prod_{\alpha\in\Phi_X^+}\langle \alpha, p\rangle dp}{\int_\Delta \prod_{\alpha\in\Phi_X^+}\langle \alpha, p\rangle dp}$$

KE metrics for Fano spherical manifolds

Theorem [D.2020]

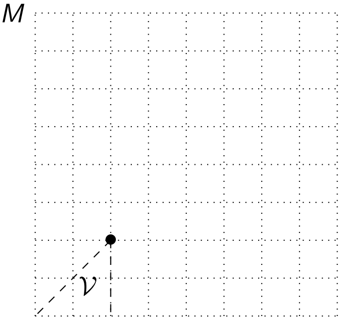
X admits a KE metric if and only if $\mathrm{Bar}(\Delta) - 2\varpi_X \in \mathrm{Relint}(-\mathcal{V}^\vee)$ where

$$\mathrm{Bar}(\Delta) = \frac{\int_\Delta p \prod_{\alpha \in \Phi_X^+} \langle \alpha, p \rangle dp}{\int_\Delta \prod_{\alpha \in \Phi_X^+} \langle \alpha, p \rangle dp}$$

Note:

1. $2\varpi_X \in \Delta$, in particular, $\mathrm{Bar}(\Delta) - 2\varpi_X \in M \otimes \mathbb{R}$
2. If $\mathcal{V} = N \otimes \mathbb{R}$ (e.g. toric case), the condition is $\mathrm{Bar}(\Delta) = 2\varpi_X$ and (as we will see later) is equivalent to vanishing of Futaki character
3. Upshot: in general, much stronger condition than in toric case
K-stability appears!
4. The measure in the integral is (strongly) related to Weyl dimension formula, will see this more precisely in Lecture 3.

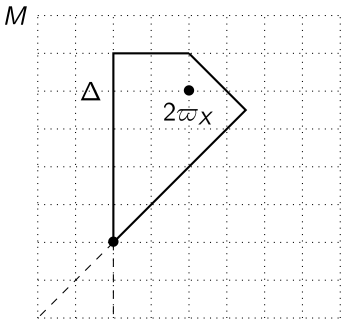
First application



wonderful compactification of $\mathrm{Sp}_4(\mathbb{C})$

Biequivariant connected reductive group compactifications are spherical:
$B \times B$ acts on $G = \frac{G \times G}{\mathrm{diag}\, G}$ with an open orbit (Bruhat's decomposition)

First application



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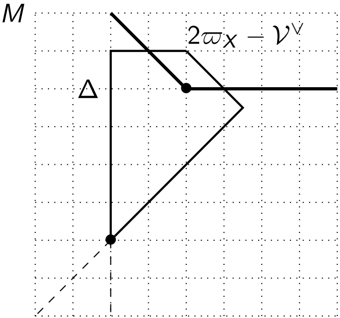

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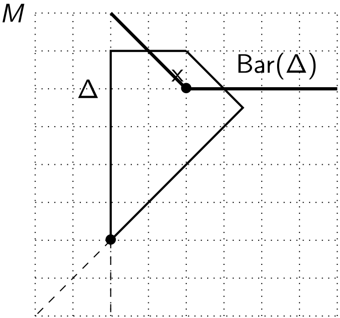

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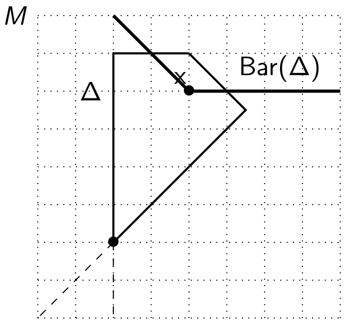
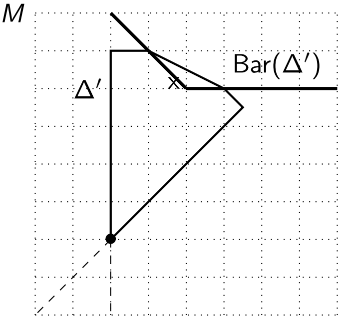


$\exists$ KE metric

Biequivariant connected reductive group compactifications are spherical:
$B \times B$ acts on $G = \frac{G \times G}{\text{diag } G}$ with an open orbit (Bruhat's decomposition)

First application



blowup of previous one: not KE

Biequivariant connected reductive group compactifications are spherical: $B \times B$ acts on $G = \frac{G \times G}{\text{diag } G}$ with an open orbit (Bruhat's decomposition)

Other applications: Greatest Ricci lower bound

- [Odaka-Okada 2013] conjectured that Picard rank one Fano manifold are K-semistable
- [Fujita 2015] two counterexamples
- [Pasquier 2009] There are infinite families of smooth and Fano (horo)spherical varieties with Picard number one, which are not homogeneous under a larger group. Their automorphism group is not reductive. In particular, by Matsushima's obstruction, they do not admit KE metrics.
- [Chi Li 2017] $GRLB(X) = 1$ iff X is K-semistable.
- For general spherical Fano manifolds, can compute $GRLB(X)$ as well with a formula involving $\mathrm{Bar}(\Delta)$ (e.g. in [D.2020] for horosymmetric, see Lecture 2 tomorrow)
- for the non-KE example in the previous slide,

$$GRLB(X) = \frac{1046175339}{1236719713} < 1$$

- for Pasquier's example, can check that $GRLB(X) < 1$ as well, hence they are K-unstable.
- Infinitely many counterexamples to Odaka and Okada's conjecture.

How to find examples

Lots of examples:

1. Spherical homogeneous space are classified, list of affine spherical homogeneous space under a simple group is reasonnably short, symmetric spaces form a large family.
2. Given a spherical homogeneous space of rank r, about as many as toric manifolds of dimension r, classified by moment polytopes
   Beware: not so easy to tell whether a polytope is the polytope of a polarized spherical variety [Cupit-Foutou, Pezzini, Van Steirteghem]
3. Much more examples than toric manifolds: infinitely many examples with dimension 1 moment polytope!
4. Homogeneous bundle construction (sometimes called parabolic induction) allows to build new examples from known examples
5. Start from a homogeneous manifold, take a subgroup of automorphism that does not act transitively, blow up some orbits
6. closure of orbits in another spherical manifold

More on examples

By dimension:

- Dimension 1: only $\mathbb{P}^1$
- Dimension 2: spherical varieties are toric
- Dimension 3: spherical varieties are T-varieties of complexity ≤ 1 (i.e. a maximal torus of the automorphism group acts with codimension (at most) one orbits)
- Higher dimensions: most spherical varieties are not T-varieties of complexity ≤ 1

The list of Fano spherical manifolds up to dimension 3 is essentially known.

By rank:

- Rank 1: next slides
- Rank 2: symmetric spaces [Ruzzi], wonderful rank two varieties [Wasserman] (not all Fano)

Horospherical: [Pasquier]

Useful reference book with lots of examples and constructions: [Timashev]
Also [Brion], [Pezzini], etc.

Examples of rank one spherical manifolds

The SL_2-varieties $\mathbb{P}^2$ and $\mathrm{Bl}_{1pt}\mathbb{P}^2$

SL_2-action extended from the natural linear action on an affine chart, three orbits: $\mathbb{P}^1$ at infinity, 0 and $\mathbb{C}^*$.
Blowup $\mathbb{P}^2$ at the fixed point 0, it gives an SL_2-homogeneous fiber bundle over $\mathbb{P}^1 = \mathrm{SL}_2 / B$ with fiber the toric variety $\mathbb{P}^1$.

Pasquier's Picard rank one examples

A determinantal variety

The SL_3^2-variety $\mathbb{P}^8 = \mathbb{P}$(non-invertible 3×3 matrices).
Orbits given by rank (1, 2 or 3).
Blowup closed orbit of rank one matrices, get a SL_3^2-homogeneous fiber bundle over $\mathbb{P}^2 \times \mathbb{P}^2$ with fiber the SL_2^2-variety $\mathbb{P}^3 = \mathbb{P}(2 \times 2$ matrices).

Classification of rank one varieties

[Akhiezer 83], [Huckleberry-Snow 82]

Classification from:

- an explicit list of *cuspidal* cases (next slide), and
- a construction from these up to blowdown:

  given X G-spherical rank 1,
  there exists $\tilde{X} \to X$ birational, G-equivariant,

  such that

  $\tilde{X} \to G/P$ G-homogeneous fiber bundle over a rational homogeneous space G/P, with fiber a cuspidal rank 1 spherical S-variety, where S Levi subgroup of P.

For polarized manifolds, rank one spherical manifolds coincide with *cohomogeneity one manifolds*: manifolds equipped with a compact Lie group action with real hypersurface orbits. These have been instrumental in the development of canonical Kähler metrics (Calabi's extremal metrics on Hirzebruch surfaces leading to Calabi's ansatz, Koiso-Sakane first examples of non-homogeneous Fano Kähler-Einstein manifolds), but mostly considered when the cuspidal case is the toric $\mathbb{P}^1$ and there are no blowdowns. In other words, mostly considered homogeneous $\mathbb{P}^1$-bundles over generalized flag manifolds.

List of cuspidal cases from [Timashev]

Table 5.10: Wonderful varieties of rank 1

No.	G	H	$H \hookrightarrow G$	$\Pi^{\min}_{G/H}$	Wonderful embedding
1	$\mathrm{SL}_2 \times \mathrm{SL}_2$	SL_2	diagonal	$\omega + \omega'$	$X = \{(x : t) \mid \det x = t^2\} \subset \mathbb{P}(\mathrm{L}_2 \oplus \Bbbk)$
2	$\mathrm{PSL}_2 \times \mathrm{PSL}_2$	PSL_2		$2\omega + 2\omega'$	$\mathbb{P}(\mathrm{L}_2)$
3	SL_n	GL_{n-1}	symmetric No. 1 	$\omega_1 + \omega_{n-1}$	$\mathbb{P}^n \times (\mathbb{P}^n)^*$
4	PSL_2	PO_2	symmetric No. 3	$4\omega_1$	$\mathbb{P}(\mathfrak{sl}_2)$
5	Sp_{2n}	$\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$	symmetric No. 4	ω_2	$\mathrm{Gr}_2(\Bbbk^{2n})$
6	Sp_{2n}	$B(\mathrm{Sp}_2) \times \mathrm{Sp}_{2n-2}$		ω_2	$\mathrm{Fl}_{1,2}(\Bbbk^{2n})$
7	SO_n	SO_{n-1}	symmetric No. 6	ω_1	$X = \{(x : t) \mid (x, x) = t^2\} \subset \mathbb{P}^n$
8	SO_n	$\mathrm{S}(\mathrm{O}_1 \times \mathrm{O}_{n-1})$		$2\omega_1$	$\mathbb{P}^{n-1}$
9	SO_{2n+1}	$\mathrm{GL}_n \rightthreetimes \bigwedge^2 \Bbbk^n$		ω_1	$X = \{(V_1, V_2) \mid V_1 \subset V_1^{\perp}\} \subset \mathrm{Fl}_{n,2n}(\Bbbk^{2n+1})$
10	Spin_7	$\mathbf{G}_2$	non-symmetric No. 10	ω_3	$X = \{(x : t) \mid (x, x) = t^2\} \subset \mathbb{P}(V(\omega_3) \oplus \Bbbk)$
11	SO_7	$\mathbf{G}_2$		$2\omega_3$	$\mathbb{P}(V(\omega_3))$
12	$\mathbf{F}_4$	$\mathbf{B}_4$	symmetric No. 17	ω_1	
13	$\mathbf{G}_2$	SL_3	non-symmetric No. 12	ω_1	$X = \{(x : t) \mid (x, x) = t^2\} \subset \mathbb{P}(V(\omega_1) \oplus \Bbbk)$
14	$\mathbf{G}_2$	$N(\mathrm{SL}_3)$		$2\omega_1$	$\mathbb{P}(V(\omega_1))$
15	$\mathbf{G}_2$	$\mathrm{GL}_2 \rightthreetimes (\Bbbk \oplus \Bbbk^2) \otimes \bigwedge^2 \Bbbk^2$		$\omega_2 - \omega_1$	