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De l'homotopie rationnelle vers l'homotopie pluripotentielle

From rational homotopy theory
towards pluripotential homotopy theory

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Foreword

This text focuses on a recent result proved in [PSZ25]. The main goal is to understand the methods used to establish strong formality for complete smooth toric varieties and compact homogeneous Kähler manifolds.

The text is structured as follows : the first section provides the necessary background in rational homotopy theory. The second introduces equivariant cohomology and discusses some of its main properties. The third adapts these constructions to the bigraded setting and is devoted to the proof of the two main results.

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Contents

1. Some background in rational homotopy theory

In this section we give an outline of Sullivan's foundational work on rational homotopy theory and its subsequent development by Deligne, Griffiths and Morgan following [Sul74, DGMS75] and [Sul77]. Standard references for the subject are the textbooks [GM81, FHT01]. A more recent overview of the current state of the art is given in [Suc23].

Let k be a field that contains $\mathbb{Q}$. We shall mainly focus on $k = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. A *commutative differential graded algebra* over k, or CDGA for short, is a k-algebra $\mathscr{A}$, which is graded, meaning that there exist vector spaces $\mathscr{A}^i$ for $i \in \mathbb{N}$ such that

$$\mathscr{A} = \bigoplus_{i \geq 0} \mathscr{A}^i$$

that are compatible with the algebra structure, i.e. $\mathscr{A}^i \cdot \mathscr{A}^j \subset \mathscr{A}^{i+j}$. The algebra $\mathscr{A}$ is furthermore equipped with a graded map $d : \mathscr{A}^\bullet \to \mathscr{A}^{\bullet+1}$ of degree $+1$, called the *differential*, and satisfies

(i) graded-commutativity, meaning

$$x \cdot y = (-1)^{|x|\cdot|y|} y \cdot x,$$

where $|x| = k$ means $x \in \mathscr{A}^k$.

(ii) d being a derivation, meaning

$$d(x \cdot y) = dx \cdot y + (-1)^{|x|} x \cdot dy.$$

(iii) d being a cohomological differential, meaning

$$d^2 = 0.$$

Example 1.2. The main examples of CDGAs to have in mind would be the following :

(i) For a smooth manifold X, consider the $\mathscr{C}^\infty$ differential forms over X, together with the exterior derivative $(A^\bullet(X),\ d)$;
(ii) The de Rham cohomology $(H^\bullet_{dR}(X, k),\ d = 0)$. Obviously, in this case k is either $\mathbb{R}$ or $\mathbb{C}$;
(iii) If X is a CW-complex, one also considers the singular cohomology $(H^\bullet(X, k),\ d = 0)$.
(iv) Given a CDGA $(\mathscr{A}^\bullet,\ d)$, the cohomology ring $(H^\bullet(\mathscr{A}),\ d = 0)$ is again a CDGA.

Now a morphism of CDGAs is a k-algebra morphism that preserves these structures.

Definition 1.3. Let $(\mathscr{A}, d_{\mathscr{A}})$ and $(\mathscr{B}, d_{\mathscr{B}})$ be two CDGAs. Then $f : \mathscr{A} \to \mathscr{B}$ is said to be a *CDGA morphism* if it is a k-algebra morphism that

(i) preserves the grading, meaning that $f(\mathscr{A}^k) \subset \mathscr{B}^k$, for any natural number k;
(ii) commutes with the differentials, meaning $f \circ d_{\mathscr{A}} = d_{\mathscr{B}} \circ f$.

Remark 1.4. The second condition implies that a CDGA morphism f sends closed (resp. exact) forms to closed (resp. exact) forms. In particular, such a morphism induces a well defined CDGA morphism in cohomology $H(f) : H^\bullet(\mathscr{A}) \to H^\bullet(\mathscr{B})$.

Definition 1.5. A CDGA morphism $f : \mathscr{A} \to \mathscr{B}$ is a *quasi-isomorphism*, if $H(f)$ is an isomorphism.

1.1. **Elementary extensions. Sullivan Algebras.** Given a vector space V, one can consider an associated CDGA by choosing a degree for the vectors lying in V, formally adding all the products of vectors, empty products included, and quotienting by the graded-commutativity relations.

Definition 1.6. If the chosen degree is $n \geq 0$, such a CDGA is called the *free algebra on V in degree n*, and is denoted by $\Lambda_n(V)$. The differential of such a CDGA, unless mentioned otherwise, is the null map.

Example 1.7. Let $V = \mathbb{R}x$ be a 1-dimensional $\mathbb{R}$-vector space. The free algebra on V in degree 1 is then $(\mathbb{R} \oplus \mathbb{R}x,\ d = 0)$. Indeed, the elements $x \cdot x = -x \cdot x$ are null by graded-commutativity, hence all the superior x^k are null as well.

On the other hand, the free algebra on V in degree 2 can be written

$$\Lambda_2(V) = \bigoplus_{k \geq 0} \Lambda^{2k},$$

where $\Lambda^{2k} = \mathbb{R}x^k$ are the elements of degree $2k$ of the CDGA.

It is easy to convince oneself that this construction is functorial. Indeed, if $f : V \to W$ is a map between two vector spaces, then there is a unique CDGA morphism that equals f when restricted to V. Indeed, this map is determined by the images of its CDGA generators, which are precisely the vectors in V. This discussion proves the following

Lemma 1.8. *For $n \geq 0$, $\Lambda_n : \mathbf{Vect}_k \to \mathbf{CDGA}$ is a functor.*

More generally, if V is graded (over $\mathbb{N}$) vector space

$$V = \bigoplus_{i \geq 0} V^i$$

one can construct a CDGA on V that takes into account its grading.

Definition 1.9. If V is a graded vector space, the *free algebra on V* is defined to be

$$\Lambda(V) = (\bigotimes_{i \geq 0} \Lambda_i(V^i))/I,$$

where I is the ideal of the graded-commutativity relations.

One then has a result similar to the one of Lemma 1.8, essentially motivated by the same argument.

Lemma 1.10. *The free algebra operation $\Lambda : \mathbf{GrVect}_k^{\geq 0} \to \mathbf{CDGA}$ is a functor.*

Now to get an inverse operation, one needs to impose restrictions on the target category. For instance, Λ is an equivalence of categories, when corestricted to its essential image, i.e. the category of free CDGAs.

A fundamental construction that will be used throughout this section is the notion of *elementary extension*. Let $\mathscr{A}$ be a CDGA.

Definition 1.11. An *elementary extension* (or a *Hirsch extension*) over $(\mathscr{A},\ d_{\mathscr{A}})$ is a CDGA of the form $(\mathscr{B} = \mathscr{A} \otimes \Lambda(V),\ d_{\mathscr{B}})$ satisfying $d_{\mathscr{B}}|_{\mathscr{A}} = d_{\mathscr{A}}$, $d_{\mathscr{B}}(V) \subset \mathscr{A}$ and V being a finite-dimensional positively graded vector space. An elementary extension over $\mathscr{A}$ will be noted $\mathscr{A} \otimes_d \Lambda(V)$, to emphasize that the aforementioned properties on the differential are implied.

Informally speaking, elementary extensions are the algebraic analogue of principal fibrations. A natural line of study is then to emulate the theory of Postnikov towers, i.e. by looking at *sufficiently nice* successive fibrations. One is then naturally brought to the following definition, which is due to Sullivan [DGMS75, Sul77].

Definition 1.12. A *Sullivan algebra* $\mathscr{M}$ is a differential algebra that can be written as a sequence of elementary extensions starting with k, called a *series* for $\mathscr{M}$, i.e. there exist $(\mathscr{M}_i)_{i\geq 1}$ with

$$k = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset \dots, \text{ such that } \bigcup_{i\geq 0} \mathscr{M}_i = \mathscr{M},$$

and such that for all i, $\mathscr{M}_i \subset \mathscr{M}_{i+1}$ is an elementary extension. In particular, a Sullivan algebra is a free algebra.

Remark 1.13. The step one differential algebra $\mathscr{M}_1$ needs to be an elementary extension over k, thus of the form $k \otimes_d \Lambda(V)$. One has, by definition, $d(V) \subset k$, and necessarily $d = 0$ for degree reasons.

Remark 1.14. It is also useful to note that there is no canonical way of constructing a series for a Sullivan algebra, unless some other assumptions are made. In general, series are far from being unique.

Example 1.15. Both free algebras already seen in Example 1.7 are Sullivan algebras, two series being stationary, equal to the CDGAs themselves after step one.

Any sequence of elementary extensions over these CDGAs is a Sullivan algebra as well.

An example of a CDGA which is not a Sullivan algebra would be

$$\mathscr{N} = (\Lambda_1(x, y, z),\ dx = y \cdot z,\ dy = z \cdot x,\ dz = x \cdot y).$$

Indeed, one proves that there is no series for which d can satisfy $d_{\mathscr{N}_i}(V_i) \subset \mathscr{N}_{i-1}$, where $\mathscr{N}_i = \mathscr{N}_{i-1} \otimes_d \Lambda(V_i)$. There is indeed an underlying property that a Sullivan algebra with *decomposable* differential needs to satisfy that does not hold for $\mathscr{N}$, which shall be discussed further.

1.2. **Minimality. Minimal models.**

Definition 1.16. Let $(\Lambda V, d)$ be a free CDGA. There is a natural *word-length grading* given by

$$\Lambda V = \bigoplus_{r\geq 0} \Lambda^r V$$

where $\Lambda^r V$ is generated by all the monomials of degree r in elements of V.

Example 1.17. For instance, if $V = k \cdot x$, then

$$\Lambda_1(x) = \bigoplus_{r\geq 0} \Lambda^r V = k \oplus \bigoplus_{n\geq 0} \Lambda^{2n+1} V = k \oplus \bigoplus_{n\geq 0} k \cdot x^{2n+1}$$

since all the positive even powers of x are zero as a consequence of graded-commutativity.

Definition 1.18. Let $(\Lambda V, d)$ be a free CDGA. The differential d is called *decomposable* if for every $x \in \Lambda V$, one has $dx \in \Lambda^+ V \cdot \Lambda^+ V$. If ΛV is moreover a Sullivan algebra, then it is said to be *minimal*.

Example 1.19. The elementary extension $(\Lambda_2(x)\otimes_d \Lambda_1(y),\ dy = x)$ is non-minimal. On the other hand, $(\Lambda_2(x)\otimes_d \Lambda_3(y),\ dy = x^2)$ is minimal. One remarks that starting with the step 2 differential algebra $\mathscr{M}_2$ in a series of a Sullivan algebra, the condition of d being decomposable is no longer trivially satisfied.

Suppose now that $\mathscr{A}$ is a CDGA with $\mathscr{A}^0 = k$. Notice that this is the case for ΛV whenever V is positively (>0) graded. It is then natural to define an *augmentation ideal*

$$A(\mathscr{A}) = \bigoplus_{i>0} \mathscr{A}^i$$

and a graded *space of indecomposables*

$$I(\mathscr{A}) = A(\mathscr{A})\Big/\big(A(\mathscr{A})\cdot A(\mathscr{A})\big).$$

These are again functors.

Remark 1.20. Under this assumption, it is easy to check that $I(\Lambda(V)) \cong V$, provided grading is preserved. On the other hand, for a given CDGA $\mathscr{A}$, $\Lambda(I(\mathscr{A})) \cong \mathscr{A}$ if and only if $\mathscr{A}$ is free.

Now let $(\mathscr{M}, d)$ be a finitely generated free minimal algebra with $\mathscr{M}^0 = k$. We can form the graded vector space dual to the indecomposables

$$\mathfrak{g} = \bigoplus_{i\geq 1} \mathfrak{g}_i, \text{ where } \mathfrak{g}_i = \big(I(\mathscr{M})^i\big)^\vee.$$

The differential d being decomposable, it maps indecomposables to tensors of length at least 2

$$d : I(\mathscr{M}) \to \bigoplus_{k\geq 2} I(\mathscr{M})^{\otimes k}.$$

Consider then, for $i, j \geq 1$, the projection of d onto length 2 tensors, of degree i and j respectively, to obtain a map $d^{i,j} : I(\mathscr{M})^{i+j-1} \to I(\mathscr{M})^i \otimes I(\mathscr{M})^j$. Dualizing yields

$$\begin{aligned}[\,\cdot\,,\cdot\,] : \mathfrak{g}_i \otimes \mathfrak{g}_j &\to \mathfrak{g}_{i+j-1}\\ \xi\otimes\eta &\mapsto [\xi,\eta],\end{aligned}$$

where, if $v \in I(\mathscr{M})^{i+j-1}$ then $[\xi\,,\eta](v) = (\xi\otimes\eta)(d^{i,j}v)$. The following lemma should render our notation choices clearer.

Lemma 1.21. *The bracket $[\,\cdot\,,\cdot\,]$ defined above satisfies, for all homogeneous $\xi \in \mathfrak{g}_i$, $\eta\in\mathfrak{g}_j$, $\zeta\in\mathfrak{g}_k$*

(Graded symmetry) $$[\xi,\eta] = (-1)^{ij}[\eta,\xi],$$

(Graded Jacobi) $$(-1)^{ik}\big[[\xi,\eta],\zeta\big] + (-1)^{ji}\big[[\eta,\zeta],\xi\big] + (-1)^{kj}\big[[\zeta,\xi],\eta\big] = 0.$$

Proof. The graded symmetry of the bracket is a direct consequence of the graded commutativity on $\mathscr{M}$. Indeed,

$$[\xi,\eta](v) = (\xi\otimes\eta)(d^{i,j}v) = (\eta\otimes\xi)\big((-1)^{ij}d^{j,i}v\big) = (-1)^{ij}[\eta,\xi](v).$$

The graded Jacobi relation on $\mathfrak{g}$ can be seen as a consequence of the fact that there is a graded Lie structure on $\mathfrak{g}$ modulo a degree shift of the algebra. See [FHT01, p. 294] for a discussion on the *homotopy Lie algebra* of ΛV. □

However, this bracket does not define a graded Lie algebra structure on $\mathfrak{g}$ itself. It is neither graded antisymmetric nor degree-preserving, in the sense that one does not have

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

Nevertheless, it behaves much like a Lie bracket, provided one accounts for the appropriate sign conventions. For our purposes, we shall refer to such objects as *Lie-like algebras*. We stress that this terminology is ad hoc and is not part of the standard nomenclature.

Now, let $\mathscr{A}$ be a free algebra as above and V be a k-vector space of finite dimension. A degree n elementary extension $\mathscr{A} \to \mathscr{A} \otimes_d \Lambda_n(V)$ with d decomposable corresponds dually to a short exact sequence of Lie-like algebras

$$0 \longrightarrow V^\vee \longrightarrow \mathfrak{g}_{\mathscr{A} \otimes_d \Lambda_n V} \longrightarrow \mathfrak{g}_{\mathscr{A}} \longrightarrow 0$$

with $[v^\vee, \xi] = 0$ for $v^\vee \in V^\vee$ and $\xi \in \mathfrak{g}_{\mathscr{A} \otimes_d \Lambda_n V}$. This is simply because, if $x \in \mathscr{A} \otimes_d \Lambda_n V$ and ξ is of degree i, then $(v^\vee \otimes \xi)(d^{n,i}x) = 0$, as $d^{n,i}x$ has to be in $I(\mathscr{A})^n \otimes I(\mathscr{A})^i$.

If we now suppose $\mathscr{M}$ to be minimal of series $k \to \mathscr{M}_1 \to \mathscr{M}_2 \to \ldots$, let $\mathscr{L}_i = I(\mathscr{M}_i)^\vee$. Dually, this gives rise to a lower central series of Lie-like algebras

$$\ldots \longrightarrow \mathscr{L}_3 \longrightarrow \mathscr{L}_2 \longrightarrow \mathscr{L}_1 \longrightarrow 0$$

and inductively, each of the $\mathscr{L}_i$, and thus $\mathfrak{g}$ itself, is nilpotent in the sense that

$$[\mathfrak{g}, [\mathfrak{g}, [\ldots, [\mathfrak{g}, \mathfrak{g}] \ldots]]] = 0$$

after a finite number of iterations, having assumed $\mathfrak{g}$ of finite type. We have therefore proven

Lemma 1.22. *Let $(\mathscr{M}, d)$ be a free algebra of finite type with decomposable d. If $\mathscr{M}$ is moreover a Sullivan algebra, and therefore minimal, then the dual to the indecomposables $\mathfrak{g}$ is a Lie-like nilpotent algebra.*

Example 1.23. Recall the algebra $\mathscr{N} = \Lambda_1(x, y, z)$ from Example 1.15. Since the indecomposables are concentrated in degree 1, $\mathfrak{g}$ is actually a genuine Lie algebra, generated by $x^\vee, y^\vee$ and $z^\vee$. Let us compute $[x^\vee, y^\vee]$.

By definition,

$$[x^\vee, y^\vee](v) = (x^\vee \otimes y^\vee)(d^{1,1}v) = \begin{cases} 0, & \text{if } v = x \text{ or } y, \\ 1, & \text{if } v = z. \end{cases}$$

Hence $[x^\vee, y^\vee] = z^\vee$, as both linear functionals agree on the vectors of a basis. By the same argument, $[y^\vee, z^\vee] = x^\vee$ and $[z^\vee, x^\vee] = y^\vee$. This Lie algebra is clearly not nilpotent. In fact, one immediately recognizes $\mathfrak{g} \cong \mathfrak{so}(3)$, which is simple.

Since d is decomposable, this proves that $\mathscr{N}$ is not a Sullivan algebra, as it would otherwise be minimal and contradict Lemma 1.22.

Definition 1.24. A *minimal model* for $\mathscr{A}$ is a minimal algebra $\mathscr{M}_{\mathscr{A}}$, together with a quasi-isomorphism $\psi_{\mathscr{A}} : \mathscr{M}_{\mathscr{A}} \xrightarrow{\simeq} \mathscr{A}$. If X is a smooth manifold, we say that $\mathscr{M}$ is a model for X if it is a model for the de Rham algebra $A^\bullet_{dR}(X)$.

We shall show that if $\mathscr{A}$ is connected, a minimal model for $\mathscr{A}$ always exists and is moreover unique up to isomorphism.

1.3. **Homotopy theory of CDGAs.** Let us start with the following observation.

Lemma 1.25 (Lifting lemma). *Consider the diagram*

$$\begin{tikzcd} & \mathscr{A} \arrow[d, "\eta", "\simeq"'] \\ \mathscr{M} \arrow[r, "\psi"'] & \mathscr{B} \end{tikzcd}$$

where $\mathscr{M}$ is a Sullivan algebra and $\eta : \mathscr{A} \xrightarrow{\simeq} \mathscr{B}$ is a surjective quasi-isomorphism. Then there is a lift $\varphi : \mathscr{M} \to \mathscr{A}$ such that $\eta \circ \varphi = \psi$.

Proof. Write $\mathscr{M}$ as ΛV, with V being an increasing union of graded subspaces $V(k)$, for $k \geq 0$, such that $V(k) = V(k-1) \oplus V_k$ and $d : V_k \to \Lambda V(k-1)$.

Suppose, for a proof by induction, that for a fixed k a morphism φ is constructed until the $\Lambda V(k-1)$ stage, satisfying $\eta\varphi|_{\Lambda V(k-1)} = \psi|_{\Lambda V(k-1)}$. Let $\{v_\alpha\}_\alpha$ be a basis for V_k. As $d(V_k) \subset \Lambda V(k-1)$, φdv_α is defined and is moreover closed for the reason that $d(\varphi dv_\alpha) = \varphi(d^2 v_\alpha) = 0$. Since $\eta\varphi dv_\alpha = \psi dv_\alpha = d\psi v_\alpha$ is also closed and since η is a surjective quasi-isomorphism, there exist $a_\alpha \in \mathscr{A}$ such that $da_\alpha = \varphi dv_\alpha$ and $\eta a_\alpha = \psi v_\alpha$. Extend φ over $\Lambda(V(k-1) \oplus V_k)$ by setting $\varphi(v_\alpha) = a_\alpha$ for every α. A routine check shows that $\varphi : \Lambda V(k) \to \mathscr{A}$ is a well-defined CDGA morphism.

The induction begins at $V(-1) = 0$, since any CDGA morphism necessarily satisfies $\varphi(1) = 1$. □

One can think of a *homotopy* of CDGAs as a morphism towards an extension by the free algebra $\Lambda(t, dt)$, where t is of degree 0 and consequently dt of degree 1. It is the algebra of forms on $[0, 1]$.

Definition 1.26. Let $\mathscr{A}, \mathscr{B}$ be two CDGAs and $f, g : \mathscr{A} \to \mathscr{B}$ two morphisms. A *homotopy* from f to g is a morphism

$$H : \mathscr{A} \to \mathscr{B} \otimes \Lambda(t, dt)$$

such that $H|_{t=0} = f$ and $H|_{t=1} = g$. Restrictions of homotopies to $t = 0, 1$ correspond to evaluations of forms at these values for t, and they automatically imply $dt = 0$. When a homotopy exists between f and g, we shall say they are homotopic and note it by $f \simeq g$.

Homotopic morphisms have the merit to induce the same morphisms in cohomology.

Proposition 1.27. *If $\mathscr{M}, \mathscr{A}$ are CDGAs and $f \simeq g : \mathscr{M} \to \mathscr{A}$, then $H(f) = H(g)$.*

Proof. An element $u \in \Lambda(t, dt)$ can be uniquely written as

$$u = u_0 + u_1 t + u_2 dt + x + dy,$$

where x, y are in the ideal $I = t(1-t) \cdot \Lambda(t) \subset \Lambda(t)$ and the u_i are simply constants. There is therefore a direct sum decomposition

$$\mathscr{A} \otimes \Lambda(t, dt) = \mathscr{A} \otimes (k \oplus k \cdot t \oplus k \cdot dt) \,\oplus\, \mathscr{A} \otimes \Lambda(I \oplus d(I)).$$

Consider now a homotopy $H : f \simeq g$. It follows that there is a linear map $h : \mathscr{M} \to \mathscr{A}$ of degree -1 such that

$$H(z) = f(z) + (g(z) - f(z))t + h(z)dt + \Omega(z)$$

with $\Omega(z) \in \mathscr{A} \otimes \Lambda(I \oplus d(I))$ for all z. One has

$$H(dz) = f(dz) + (g(dz) - f(dz))t + h(dz)dt + \Omega(dz),$$
$$dH(z) = f(dz) + (g(dz) - f(dz))t + (-1)^{|z|}(g(z) - f(z))dt + dh(z)dt + d\Omega(z).$$

Notice that, since H is assumed to be a CDGA morphism, $H \circ d = d \circ H$. The elements in dt have to agree, whence

$$g(z) - f(z) = (-1)^{|z|}(h(dz) - dh(z)).$$

Now, when z is a cocycle, $g(z) - f(z) = (-1)^{|z|+1}dh(z)$ is exact, which is precisely saying that f and g agree in cohomology. □

Proposition 1.28. *Let $\mathscr{M}$ be a Sullivan algebra and $\mathscr{A}$ be any CDGA. Being homotopic is then an equivalence relation in the set $\mathrm{Hom}_{\mathbf{CDGA}}(\mathscr{M}, \mathscr{A})$.*

Proof. Consider $f, g : \mathscr{M} \to \mathscr{A}$. Reflexivity is trivially satisfied, since $f \simeq f$ by the constant homotopy map $H = f \otimes 1 : \mathscr{M} \to \mathscr{A} \otimes \Lambda(t, dt)$.

Symmetry is also easily checked, by remarking that if $H : f \simeq g$, then composing with the automorphism $t \mapsto 1 - t$ on $\Lambda(t, dt)$ produces a homotopy $H' : g \simeq f$.

The main difficulty is to prove transitivity. Suppose $f, g, h : \mathscr{M} \to \mathscr{A}$ are morphisms such that there exist homotopies

$$H : \mathscr{M} \to \mathscr{A} \otimes \Lambda(t_1, dt_1), \qquad J : \mathscr{M} \to \mathscr{A} \otimes \Lambda(t_2, dt_2)$$

with $H : f \simeq g$ and $J : g \simeq h$ respectively. Topologically this corresponds to H being a homotopy on a horizontal interval $(t_1, 0) \in [0,1] \times \{0\}$ and the homotopy J on a vertical interval $(1, t_2) \in \{1\} \times [0,1]$. Consider the topological space resulting from the gluing of these two intervals. The CDGA that corresponds to this topological space is a quotient of $\Lambda(t_1, t_2, dt_1, dt_2)$ by a certain ideal, determined by the following conditions

(i) Either $t_1 = 1$ or $t_2 = 0$, implying the relation $(t_1 - 1)t_2 = 0$.
(ii) The element dt_1 is non-zero only on the horizontal interval, meaning $t_2 dt_1 = 0$.
(iii) The element dt_2 is non-zero only on the vertical interval, meaning $(t_1 - 1)dt_2 = 0$.

Let us write

$$\Xi = \Lambda(t_1, t_2, dt_1, dt_2) \Big/ \big((t_1 - 1)t_2, t_2 dt_1, (t_1 - 1)dt_2\big)$$

for the CDGA of the two glued intervals. Also write, by abuse of notation, H, J for the homotopies composed with the canonical injections $\mathscr{A} \otimes \Lambda(t_1, dt_1) \hookrightarrow \mathscr{A} \otimes \Xi$ and $\mathscr{A} \otimes \Lambda(t_2, dt_2) \hookrightarrow \mathscr{A} \otimes \Xi$ respectively. We shall now define a morphism $K : \mathscr{M} \to \mathscr{A} \otimes \Xi$ that interpolates H and J.

Let $m \in \mathscr{M}$. The algebra $\mathscr{A} \otimes \Lambda(t_1, dt_1)$ can be thought of as an $\mathscr{A}$-module being generated by $\{(t_1)^i, (t_1)^i \cdot dt_1\}_{i \geq 0}$. Then

$$H(m) = \sum_{i \in I} a_i \otimes t_1^i + b_i \otimes t_1^i dt_1$$

in $\mathscr{A} \otimes \Xi$, for a certain finite $I \subset \mathbb{N}$. Similarly,

$$J(m) = \sum_{j \in J} a'_j \otimes t_2^j + b'_j \otimes t_2^j dt_2$$

with finite $J \subset \mathbb{N}$.

The fact that $H(m)|_{t_1=1} = J(m)|_{t_2=0} = g(m)$ implies that $\sum_{i \in I} a_i = a'_0$. Set then

$$K(m) = \sum_{i \in I} a_i \otimes t_1^i + b_i \otimes t_1^i dt_1 + \sum_{j \in J \setminus \{0\}} a'_j \otimes t_2^j + \sum_{j \in J} b'_j \otimes t_2^j dt_2.$$

This provides a CDGA morphism $K : \mathscr{M} \to \mathscr{A} \otimes \Xi$. Now let

$$p : \mathscr{A} \otimes \Lambda(t_1, t_2, dt_1, dt_2) \to \mathscr{A} \otimes \Xi$$

be the canonical projection morphism. We claim this is a surjective quasi-isomorphism. One can see this by remarking that $H^\bullet(\mathscr{A} \otimes \Lambda(t_1, t_2, dt_1, dt_2)) \cong H^\bullet(\mathscr{A} \otimes \Xi) \cong H^\bullet(\mathscr{A})$. But p induces the identity on the $H^\bullet(\mathscr{A})$ factor.

Now apply Lemma 1.25 to the diagram

$$\begin{tikzcd} & \mathscr{A} \otimes \Lambda(t_1, t_2, dt_1, dt_2) \arrow[d, "p", "\simeq"'] \\ \mathscr{M} \arrow[r, "K"'] & \mathscr{A} \otimes \Xi \end{tikzcd}$$

to obtain a lift $\widetilde{K} : \mathscr{M} \to \Lambda(t_1, t_2, dt_1, dt_2)$. Compose with $r : \Lambda(t_1, t_2, dt_1, dt_2) \to \Lambda(t, dt)$ which maps $t_1, t_2 \mapsto t$ and $dt_1, dt_2 \mapsto dt$, that corresponds to the pullback of the inclusion of the diagonal $\Delta = \{t_1 = t_2\} \subset \mathbb{A}^2$. This yields a homotopy

$$(\mathrm{id} \otimes r) \circ \widetilde{K} : \mathscr{M} \to \mathscr{A} \otimes \Lambda(t, dt)$$

between f and h.

$\square$

Definition 1.29. Let $\mathscr{M}$ be a Sullivan algebra and $\mathscr{A}$ be any CDGA. We note by $[\mathscr{M}, \mathscr{A}]$ the set of homotopy classes of CDGA morphisms $\mathscr{M} \to \mathscr{A}$.

The two following lemmas will be admitted without proof. We refer the reader to [GM81, Theorem 11.5 & Lemma 11.7] for further details.

Lemma 1.30. *Let $\varphi : \mathscr{B} \to \mathscr{C}$ be a quasi-isomorphism and $\mathscr{M}$ be a Sullivan algebra. Then $\varphi_* : [\mathscr{M}, \mathscr{B}] \to [\mathscr{M}, \mathscr{C}]$ is a bijection.*

Lemma 1.31. *Suppose $\mathscr{M}$ and $\mathscr{M}'$ are two minimal algebras, together with a quasi-isomorphism $I : \mathscr{M} \to \mathscr{M}'$. Then I is an isomorphism of CDGAs.*

Theorem 1.32 (Uniqueness of the minimal model)**.** *Let $\mathscr{A}$ be a CDGA and $\rho : \mathscr{M} \to \mathscr{A}$, $\rho' : \mathscr{M}' \to \mathscr{A}$ be two minimal models for $\mathscr{A}$. Then there is an isomorphism $I : \mathscr{M} \to \mathscr{M}'$ and a homotopy $H : \rho \simeq \rho' \circ I$. Such an isomorphism is itself unique up to homotopy.*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "="] & \mathscr{A} \\ \mathscr{M} \arrow[u, "\rho"] \arrow[r, dashed, "I"'] & \mathscr{M}' . \arrow[u, "\rho'"'] \end{tikzcd}$$

A *caveat* is due before the proof of this statement.

Remark 1.33. Without additional assumptions, the diagram above is only homotopy commutative.

Proof. It suffices to apply Lemma 1.30 to the quasi-isomorphism $\rho' : \mathscr{M}' \to \mathscr{A}$. Then $\rho'_* : [\mathscr{M}, \mathscr{M}'] \to [\mathscr{M}, \mathscr{A}]$ is a bijection and it follows that there is a unique up to homotopy morphism $I : \mathscr{M} \to \mathscr{M}'$ such that $\rho' \circ I \simeq \rho$. At the cohomology level $[I] = [\rho']^{-1} \circ [\rho]$, therefore I is a quasi-isomorphism. Now apply Lemma 1.31 to conclude that I is actually an isomorphism. $\square$

Another immediate corollary of Lemma 1.30 would be the following.

Theorem 1.34. *Let $\psi_{\mathscr{A}} : \mathscr{M}_{\mathscr{A}} \to \mathscr{A}$ and $\psi_{\mathscr{B}} : \mathscr{M}_{\mathscr{B}} \to \mathscr{B}$ be minimal models. Let $f : \mathscr{A} \to \mathscr{B}$ be a CDGA morphism. Then there is a map $\hat{f} : \mathscr{M}_{\mathscr{A}} \to \mathscr{M}_{\mathscr{B}}$ such that $\psi_{\mathscr{B}} \circ \hat{f} \simeq f \circ \psi_{\mathscr{A}}$. Such a map $\hat{f}$ is unique up to homotopy.*

Proof. It suffices to apply Lemma 1.30 to the diagram

$$\begin{tikzcd} & \mathscr{B} \\ \mathscr{M}_{\mathscr{A}} \arrow[ur, "f\circ\psi_{\mathscr{A}}"] \arrow[r, dashed, "\hat{f}"'] & \mathscr{M}_{\mathscr{B}}. \arrow[u, "\psi_{\mathscr{B}}"'] \end{tikzcd}$$

Since $\psi_{\mathscr{B}}$ is a quasi-isomorphism and $\mathscr{M}_{\mathscr{A}}$ is a minimal algebra, it ensues that

$$(\psi_{\mathscr{B}})_* : [\mathscr{M}_{\mathscr{A}}, \mathscr{M}_{\mathscr{B}}] \to [\mathscr{M}_{\mathscr{A}}, \mathscr{B}]$$

is a bijection. There is then a unique up to homotopy $\hat{f} : \mathscr{M}_{\mathscr{A}} \to \mathscr{M}_{\mathscr{B}}$ such that $f \circ \psi_{\mathscr{A}} \simeq \psi_{\mathscr{B}} \circ \hat{f}$. Note that Remark 1.33 also holds here. ☐

1.4. **Relative Sullivan algebras.** In this section we discuss the relative case of the (minimal) Sullivan algebras introduced in sections 1.1 and 1.2. These turn out to be the appropriate framework for rational homotopy theory of fibrations. Moreover, they allow us to prove existence of minimal models under minimal hypotheses.

Let $\mathscr{B}$ be a connected CDGA.

Definition 1.35. A *relative Sullivan algebra* $\mathscr{R}$ (over $\mathscr{B}$) is a differential algebra that can be written as a sequence of elementary extensions over $\mathscr{B}$, i.e. there exist $(\mathscr{R}_i)_{i\geq 1}$ with

$$\mathscr{B} = \mathscr{R}_0 \subset \mathscr{R}_1 \subset \mathscr{R}_2 \subset \ldots, \text{ such that } \bigcup_{i\geq 0} \mathscr{R}_i = \mathscr{R},$$

and such that for all i, $\mathscr{R}_i \subset \mathscr{R}_{i+1}$ is an elementary extension.

Equivalently, $\mathscr{R}$ is a CDGA of the form $(\mathscr{B} \otimes \Lambda V, d)$, such that there is a sequence of positively graded subspaces $V(0) \subset V(1) \subset \ldots$, with $\bigcup_{k\geq 0} V(k) = V$, and such that

$$d : V(0) \to \mathscr{B} \quad \text{and} \quad d : V(k) \to \mathscr{B} \otimes \Lambda\big(V(k-1)\big),\ k \geq 1.$$

The CDGA $\mathscr{B}$ in this case is called the *base*.

Remark 1.36. A Sullivan algebra is also a relative Sullivan algebra with $\mathscr{B} = k$.

Now consider a CDGA morphism $\varphi : \mathscr{B} \to \mathscr{C}$ with connected $\mathscr{B}$.

Definition 1.37. A *Sullivan model* for φ is a quasi-isomorphism

$$m : \mathscr{B} \otimes \Lambda V \xrightarrow{\simeq} \mathscr{C}$$

with $\mathscr{B} \otimes \Lambda V$ a relative Sullivan algebra and $m|_{\mathscr{B}} = \varphi$. We shall also say, by abuse of language, that $\mathscr{B} \otimes \Lambda V$ is a Sullivan model for φ.

It is worth noting that a Sullivan model m for φ does not always exist. For instance, its existence implies that $H^0(\mathscr{B} \otimes \Lambda V) \cong H^0(\mathscr{C})$, and since $\mathscr{B}$ is connected and V is positively graded, $\mathscr{B} \otimes \Lambda V$ is also connected. Thus for m to exist, $\mathscr{C}$ must be connected.

Additionally, if i denotes the canonical injection $b \mapsto b \otimes 1$, note that $H^1(i)$ is injective. Indeed, if a degree one element b becomes a coboundary dy in $\mathscr{B} \otimes \Lambda V$, then y is in $(\mathscr{B} \otimes \Lambda V)^0 = \mathscr{B}^0$, and thus it was a coboundary from the start. But then, by definition $m \circ i = \varphi$, thus $H^1(m) \circ H^1(i) = H^1(\varphi)$ is injective necessarily.

Therefore, a necessary condition for m to exist is that $\mathscr{C}$ be connected and that $H^1(\varphi)$ be injective. This condition also turns out to be sufficient.

Proposition 1.38. *Let $\varphi : \mathscr{B} \to \mathscr{C}$ be a morphism of connected CDGAs, such that $H^1(\varphi)$ is injective. Then there exists a Sullivan model $m : \mathscr{B} \otimes \Lambda V \xrightarrow{\simeq} \mathscr{C}$ for φ.*

Proof. We shall construct a Sullivan model $\mathscr{B} \otimes \Lambda V \xrightarrow{\simeq} \mathscr{C}$, with $V = \bigoplus_{k\geq 0} V_k$, by induction over k.

Consider a vector space $V_0 \subset H^+(\mathscr{C})$ such that $V_0 \oplus H(\varphi)(H^+(\mathscr{B})) = H^+(\mathscr{C})$. Choose a basis $\{v_\alpha\}_\alpha$ of V_0, as well as representatives for these vectors in $\mathscr{C}$. Note by $\psi : V_0 \to \mathscr{C}$ the morphism that sends v_α to its corresponding chosen representative. Since both $\mathscr{B}$ and $\mathscr{C}$ are connected, it is clear that the CDGA morphism

$$m_0 = \varphi \otimes \Lambda(\psi) : (\mathscr{B} \otimes \Lambda V_0,\ d_0 = d_{\mathscr{B}} \otimes 0) \to (\mathscr{C},\ d_{\mathscr{C}})$$

induces a surjection $H(m_0)$. Note that $H^1(m_0)$ splits as $H^1(\varphi) \oplus (V_0^1 \hookrightarrow H^1(\mathscr{C}))$, the latter being canonically injective. It follows that $\operatorname{Ker} H^1(m_0) \cong \operatorname{Ker} H^1(\varphi) = 0$. Injectivity in all degrees is not preserved in general. For instance, if V_0 contains a positive even degree vector w, then the powers w^k in ΛV_0 need not vanish. The cohomology classes $[w^k]$ may then obstruct the injectivity of $H^{k|w|}(m_0)$.

Fix $k \in \mathbb{N}$ and suppose we have constructed an extension of m_0

$$m_k : (\mathscr{B} \otimes \Lambda W(k), d_k) := \Big(\mathscr{B} \otimes \Lambda\Big(\bigoplus_{i=0}^{k} V_i\Big), d_k\Big) \to (\mathscr{C}, d_{\mathscr{C}})$$

which induces a surjection in cohomology. Choose a basis $\{[z_\beta]\}_\beta$ for $\operatorname{Ker} H(m_k)$, as well as representatives $\{z_\beta\}_\beta$ in $\mathscr{B} \otimes \Lambda W(k)$. Let V_{k+1} be a vector space with a basis $\{v_\beta\}_\beta$ bijective to the basis $\{z_\beta\}_\beta$, such that $\deg v_\beta = \deg z_\beta - 1$. Define a derivation d_{k+1} on $(\mathscr{B} \otimes \Lambda W(k)) \otimes_d \Lambda V_{k+1}$, by setting $d_{k+1}(v_\beta) = z_\beta$ for every β, and $d_{k+1}|_{\mathscr{B}\otimes\Lambda W(k)} = d_k$. Now for $x, y \in (\mathscr{B} \otimes \Lambda W) \otimes_d \Lambda V_{k+1}$, the identity

$$\begin{aligned}
(d_{k+1})^2(x \cdot y) &= d_{k+1}\big(d_{k+1}(x) \cdot y + (-1)^{|x|} x \cdot d_{k+1}(y)\big) \\
&= (d_{k+1})^2(x) \cdot y + (-1)^{|x|+1} d_{k+1}(x) \cdot d_{k+1}(y) \\
&\qquad + (-1)^{|x|} d_{k+1}(x) \cdot d_{k+1}(y) + x \cdot (d_{k+1})^2(y) \\
&= (d_{k+1})^2(x) \cdot y + x \cdot (d_{k+1})^2(y)
\end{aligned}$$

holds. Since $(d_{k+1})^2(v_\beta) = d_k(z_\beta) = 0$ and since $(d_{k+1})^2|_{\mathscr{B}\otimes\Lambda W(k)} = (d_k)^2 = 0$, it follows at once that $(d_{k+1})^2 = 0$. The map d_{k+1} is thus a CDGA differential.

Now in order to construct $m_{k+1} : (\mathscr{B} \otimes \Lambda W(k)) \otimes_d V_{k+1} \to \mathscr{C}$, recall that $H(m_k)(z_\beta) = 0$, since the $[z_\beta]$ were chosen in $\operatorname{Ker} H(m_k)$. It follows that there exist $\{a_\beta\}_\beta$ in $\mathscr{C}$ such that $d_{\mathscr{C}} a_\beta = m_k(z_\beta)$ for all β. Choose such $\{a_\beta\}_\beta$ and define a graded algebra morphism m_{k+1}, by setting $m_{k+1}(v_\beta) = a_\beta$ and $m_{k+1}|_{\mathscr{B}\otimes\Lambda W(k)} = m_k$. Then $m_{k+1} d_{k+1} = d_{\mathscr{C}} m_{k+1}$, therefore it is a CDGA morphism. This completes the construction of a CDGA morphism

$$m : (\mathscr{B} \otimes \Lambda V, d') = \Big(\mathscr{B} \otimes \Lambda\Big(\bigoplus_{i=0}^{\infty} V_i\Big), d'\Big) \to (\mathscr{C}, d_{\mathscr{C}}),$$

such that $m|_{\mathscr{B}\otimes\Lambda W(k)} = m_k$ for all k.

It induces a surjection in cohomology, since $H(m)|_{H^\bullet(\mathscr{B}\otimes\Lambda V_0)} = H(m|_{\mathscr{B}\otimes\Lambda V_0}) = H(m_0)$ is already surjective. It is moreover injective in cohomology, because if z is a representative of a cohomology class in $H^\bullet(\mathscr{B}\otimes\Lambda V)$ such that $H(m)([z]) = 0$, then $H(m_k)([z]) = 0$

for a large enough k. By construction z is a coboundary in $\mathscr{B} \otimes \Lambda W(k+1) \subset \mathscr{B} \otimes \Lambda V$, whence the injectivity of $H(m)$. It follows that m is a quasi-isomorphism.

It remains to check that $\mathscr{B} \otimes \Lambda V$ is a relative Sullivan algebra. By construction, one has a sequence of successive elementary extensions $(\mathscr{B} \otimes \Lambda W(k))_{k\geq 0}$ whose union is $\mathscr{B} \otimes \Lambda V$. One needs to verify that the $W(k)$ are positively graded.

This is certainly true for $V_0 = W(0)$, since $V_0 \subset H^+(\mathscr{C})$ by construction. Recall that $H^1(m_0)$ is injective. Suppose, for a proof by induction, that for a fixed $k \in \mathbb{N}$ one has $W(k)$ concentrated in degrees ≥ 1 and $H^1(m_k)$ injective. That $H^0(m_k)$ is injective, and in fact bijective, follows from this assumption and Künneth's formula, as

$$H^0(\mathscr{B} \otimes \Lambda W(k)) \cong H^0(\mathscr{B}) \otimes H^0(\Lambda W(k)) = \text{ground field}$$

since $W(k)$ is positively graded, whence both $\mathscr{B}$ and $\Lambda W(k)$ are connected. Thus $H^0(m_k)$ is the identity map of the ground field, since $m_k(1) = 1$. It follows that $\operatorname{Ker} H(m_k)$ is concentrated in degrees ≥ 2, thus V_{k+1}, and consequently $W(k+1)$, are concentrated in degrees ≥ 1. To prove injectivity of $H^1(m_{k+1})$, consider the direct sum decomposition

$$(\mathscr{B} \otimes \Lambda W(k+1))^1 = \mathscr{B}^1 \oplus \mathscr{B}^0 \otimes V_0^1 \oplus \ldots \oplus \mathscr{B}^0 \otimes V_{k+1}^1,$$

and let z be a representative of a class $[z] \in \operatorname{Ker} H^1(m_{k+1})$. Write

$$z = b + v_0 + \ldots + v_k + v_{k+1}, \text{ with } b \in \mathscr{B}^1 \text{ and } v_i \in \mathscr{B}^0 \otimes V_i^1 \text{ for all } i.$$

By definition, $d_{k+1}z = 0$, hence $d_{k+1}(v_{k+1})$ is in $d_k(\mathscr{B} \otimes \Lambda W(k))$. Now, if the chosen basis for V_{k+1} is $\{v_\beta\}_\beta$, then $v_{k+1} = \sum_\beta c_\beta v_\beta$. By construction, $d_{k+1}(v_{k+1}) = \sum_\beta c_\beta z_\beta$ in $\mathscr{B} \otimes \Lambda W(k)$. Saying that it is moreover an element of $d_k(\mathscr{B} \otimes \Lambda W(k))$ amounts to saying that after taking the cohomology, one has $\sum_\beta c_\beta [z_\beta] = 0$. But the $\{[z_\beta]\}_\beta$ are free, hence all the c_β are null. Hence $v_{k+1} = 0$ and $z \in \mathscr{B} \otimes \Lambda W(k)$. But $m_{k+1}|_{\mathscr{B}\otimes\Lambda W(k)} = m_k$ and $H^1(m_k)$ is injective by the induction hypothesis. It follows that z is a coboundary already in $\mathscr{B} \otimes \Lambda W(k)$. By the way in which the sequence $(\mathscr{B} \otimes \Lambda W(i))_{i\geq 0}$ was built, z is necessarily a coboundary in $\mathscr{B} \otimes \Lambda W(k+1)$ as well, thus the injectivity of $H^1(m_{k+1})$. $\square$

There is also a notion of *minimality* in the relative case.

Definition 1.39. A relative Sullivan algebra $(\mathscr{B} \otimes \Lambda V, d)$ is *minimal* if

$$\operatorname{Im} d \subset \mathscr{B}^+ \otimes \Lambda V + \mathscr{B} \otimes \Lambda^{\geq 2} V.$$

Remark 1.40. Note that the notation $\Lambda^k V$ does not stand for the elements of the free algebra ΛV of degree k, but for the elements of length k in vectors in V.

Definition 1.41. Let $\varphi : \mathscr{B} \to \mathscr{C}$ be a CDGA morphism with connected $\mathscr{B}$. A *minimal Sullivan model for* φ is a Sullivan model $\mathscr{B} \otimes \Lambda V \xrightarrow{\simeq} \mathscr{C}$ such that $\mathscr{B} \otimes \Lambda V$ is minimal as a relative Sullivan algebra over $\mathscr{B}$.

Morphisms from relative Sullivan algebras satisfy a similar lift property to the ones from Sullivan algebras in the absolute case (Lemma 1.25).

Lemma 1.42 (Relative lifting lemma)**.** *Suppose we have a commutative square*

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathscr{A} \arrow[d, "\eta", "\simeq"'] \\
\mathscr{B} \otimes \Lambda V \arrow[r, "\psi"'] & \mathscr{C}
\end{tikzcd}$$

such that $i : \mathscr{B} \to \mathscr{B} \otimes \Lambda V$ is the base injection and such that $\eta : \mathscr{A} \xrightarrow{\simeq} \mathscr{C}$ is a surjective quasi-isomorphism. There is then a morphism $\varphi : \mathscr{B} \otimes \Lambda V \to \mathscr{A}$ such that $\varphi \circ i = \alpha$ and $\eta \circ \varphi = \psi$. We say that φ lifts ψ and extends α.

There is also a notion of relative homotopy.

Definition 1.43. Suppose one has two morphisms $f, g : \mathscr{B} \otimes \Lambda V \to \mathscr{A}$ such that $\mathscr{B} \otimes \Lambda V$ is a relative Sullivan algebra and f, g restrict to the same morphism $\alpha : \mathscr{B} \to \mathscr{A}$. We shall say that f and g are *homotopic* $\operatorname{rel} \mathscr{B}$ if there is a morphism

$$H : \mathscr{B} \otimes \Lambda V \to \mathscr{A} \otimes \Lambda(t, dt)$$

such that $H|_{t=0} = f, H|_{t=1} = g$ and $H(b) = \alpha(b) \otimes 1$ for $b \in \mathscr{B}$. The morphism H is called a *relative homotopy from* f to g, and we write $f \simeq g \operatorname{rel} \mathscr{B}$ to say that such an H exists.

Again, by a similar argument to the one in the absolute case, one concludes that relative homotopy is an equivalence relation over the set of morphisms from a relative Sullivan algebra $\mathscr{B} \otimes \Lambda V \to \mathscr{A}$ extending $\alpha : \mathscr{B} \to \mathscr{A}$.

Definition 1.44. Let $\mathscr{B} \otimes \Lambda V$ be a relative Sullivan algebra and $\mathscr{A}$ be a CDGA with a morphism $\alpha : \mathscr{B} \to \mathscr{A}$. We note by $[\mathscr{B} \otimes \Lambda V, \mathscr{A}]_\alpha$ the set of relative to $\mathscr{B}$ homotopy classes of CDGA morphisms $\mathscr{B} \otimes \Lambda V \to \mathscr{A}$ which restrict to α on $\mathscr{B}$.

Now, suppose one has the following commutative square

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathscr{A} \arrow[d, "\eta", "\simeq"'] \\
\mathscr{B} \otimes \Lambda V \arrow[r, "\psi"'] & \mathscr{C}
\end{tikzcd}$$

such that $i : \mathscr{B} \to \mathscr{B} \otimes \Lambda V$ is the base injection and such that $\eta : \mathscr{A} \xrightarrow{\simeq} \mathscr{C}$ is a quasi-isomorphism. The argument for Lemma 1.30 can also be applied in the relative case to yield

Lemma 1.45. *The pushforward $\eta_* : [\mathscr{B} \otimes \Lambda V, \mathscr{A}]_\alpha \to [\mathscr{B} \otimes \Lambda V, \mathscr{C}]_\alpha$ is a bijection. In other words, there is a unique homotopy class $\operatorname{rel} \mathscr{B}$ of morphisms $\varphi : \mathscr{B} \otimes \Lambda V \to \mathscr{A}$ such that $\varphi|_{\mathscr{B}} = \alpha$ and $\eta\varphi \simeq \psi \ \operatorname{rel}\ \mathscr{B}$.*

Definition 1.46. We say that CDGA is *contractible* if it is of the form $(\Lambda(U \oplus dU), d)$ for a certain graded vector space U.

A fundamental result about relative Sullivan algebras, for the proof of which we refer to [FHT01, Theorem 14.9], is the following.

Theorem 1.47. *Suppose $\mathscr{B} \otimes \Lambda V$ is a relative Sullivan algebra. It then admits a decomposition as a tensor product between a minimal relative Sullivan algebra and a contractible algebra. Concretely, there is a graded vector space U, such that $\mathrm{id}_{\mathscr{B}}$ extends to a CDGA isomorphism*

$$(\mathscr{B} \otimes \Lambda W) \otimes \Lambda(U \oplus dU) \xrightarrow{\cong} \mathscr{B} \otimes \Lambda V$$

where $\mathscr{B} \otimes \Lambda W$ is a minimal relative Sullivan algebra.

Recall that in the absolute case, a quasi-isomorphism between two minimal algebras $I : \mathscr{M}' \xrightarrow{\simeq} \mathscr{M}'$ is necessarily an isomorphism (Lemma 1.31). Here is an extension of that result to the relative setting.

Proposition 1.48. *Let $\eta : \mathscr{B}' \otimes \Lambda(V') \xrightarrow{\simeq} \mathscr{B} \otimes \Lambda(V)$ be a quasi-isomorphism of minimal relative Sullivan algebras. Suppose η restricts to an isomorphism on the base algebras $\eta_{\mathscr{B}} : \mathscr{B}' \xrightarrow{\cong} \mathscr{B}$. Then η is itself an isomorphism.*

Proof. Consider the inverse morphism $(\eta_{\mathscr{B}})^{-1} : \mathscr{B} \to \mathscr{B}'$ and write its composition with $\mathscr{B}' \hookrightarrow \mathscr{B}' \otimes \Lambda V'$, by abuse of notation, $(\eta_{\mathscr{B}})^{-1} : \mathscr{B} \to \mathscr{B}' \otimes \Lambda V'$. We shall construct by induction a morphism $\gamma : \mathscr{B} \otimes \Lambda V \to \mathscr{B}' \otimes \Lambda V'$ such that $\eta \circ \gamma = \mathrm{id}$.

Indeed, since the relative Sullivan algebras are supposed minimal, then $d : V \to (\mathscr{B}^+ \otimes \Lambda V) \oplus \Lambda^{\geq 2} V$. Write

$$V = \bigcup_k V(k) \text{ with } V(k) = V(k-1) \oplus V_k \text{ and } d : V_k \to \mathscr{B} \otimes \Lambda V(k-1).$$

Minimality then implies that if v is pure of degree n in V_k, then dv cannot land in $\mathscr{B}^0 \otimes \Lambda^1 V(k-1)^{n+1}$. In other words, $d(V_k^n) \subset \mathscr{B} \otimes \Lambda(V^{<n}) \otimes \Lambda(V(k-1)^n)$. Therefore, assume that γ has been constructed on $\mathscr{A} = \mathscr{B} \otimes \Lambda(V^{<n}) \otimes \Lambda(V(k-1)^n)$ and let us extend it to $\mathscr{A} \otimes \Lambda V_k^n$.

Since $\gamma \circ \eta|_{\mathscr{A}} = \mathrm{id}_{\mathscr{A}}$, there is a quasi-isomorphism between the graded vector spaces

$$\bar{\eta} : (\mathscr{B}' \otimes \Lambda V')/\gamma(A) \xrightarrow{\simeq} (\mathscr{B} \otimes \Lambda V)/A.$$

But $(\mathscr{B} \otimes \Lambda V)/\mathscr{A}$ contains no elements of degree $n-1$, therefore no coboundaries of degree n. It follows that every cocycle of degree n is the image of a cocycle of degree n in $(\mathscr{B}' \otimes \Lambda V')/\gamma(A)$. In particular, if $\{v_\alpha\}_\alpha$ is a basis of V_k^n, then there are degree n elements $x_\alpha \in \mathscr{B}' \otimes \Lambda V'$ and $a_\alpha, a'_\alpha \in \mathscr{A}$ such that

$$\eta x_\alpha = v_\alpha + a_\alpha \quad \text{and} \quad dx_\alpha = \gamma(a'_\alpha),$$

since x_α becomes a cocycle in $\mathscr{B}' \otimes \Lambda V'$. Therefore $dv_\alpha = d\eta(x_\alpha - \gamma a_\alpha) = a'_\alpha - da_\alpha$. Now extend γ to $\mathscr{A} \otimes \Lambda V_k^n$ by setting $\gamma v_\alpha = x_\alpha - \gamma a_\alpha$. This completes the induction and provides $\gamma : \mathscr{B} \otimes \Lambda \to \mathscr{B}' \otimes \Lambda V'$ such that $\eta \circ \gamma = \mathrm{id}$. Thus γ is injective.

It follows that γ is itself a quasi-isomorphism. Now the same argument applies to γ and yields $\chi : \mathscr{B}' \otimes \Lambda V' \to \mathscr{B} \otimes \Lambda V$ such that $\gamma \circ \chi = \mathrm{id}$. Thus γ is surjective, and therefore, an isomorphism. □

Lastly, recall that we have established in Proposition 1.38 an equivalent condition for the existence of a Sullivan model of a morphism. We shall now reinforce it.

Theorem 1.49. *Suppose $\mathscr{B}, \mathscr{C}$ to be connected and*

$$\varphi : \mathscr{B} \to \mathscr{C}$$

to be a CDGA morphism such that $H^1(\varphi)$ is injective. Then the morphism φ has a minimal Sullivan model

$$m : \mathscr{B} \otimes \Lambda V \xrightarrow{\simeq} \mathscr{C}.$$

Moreover, if $m' : \mathscr{B} \otimes \Lambda V' \xrightarrow{\simeq} \mathscr{C}$ is a second minimal Sullivan model for φ, then there is a CDGA isomorphism

$$\alpha : \mathscr{B} \otimes \Lambda V \xrightarrow{\cong} \mathscr{B} \otimes \Lambda V'$$

restricting to the identity on $\mathscr{B}$, such that $m'\alpha \simeq m \operatorname{rel} \mathscr{B}$.

Proof. Let $m : \mathscr{B} \otimes \Lambda V \xrightarrow{\simeq} \mathscr{C}$ be a Sullivan model for φ, provided by Proposition 1.38. By Theorem 1.47, there is a decomposition of this Sullivan model as a tensor product of a minimal relative algebra $\mathscr{B} \otimes \Lambda W$ and a contractible algebra

$$\beta : (\mathscr{B} \otimes \Lambda W) \otimes \Lambda(U \oplus dU) \xrightarrow{\cong} \mathscr{B} \otimes \Lambda V.$$

Consider the injection $i : \mathscr{B} \otimes \Lambda W \hookrightarrow (\mathscr{B} \otimes \Lambda W) \otimes \Lambda(U \oplus dU)$ given by $x \mapsto x \otimes 1$. It is clearly a quasi-isomorphism, since by Künneth's formula we can identify

$$H^\bullet((\mathscr{B} \otimes \Lambda W) \otimes \Lambda(U \oplus dU)) \cong H^\bullet(\mathscr{B} \otimes \Lambda W) \otimes H^\bullet(\Lambda(U \oplus dU)) \cong H^\bullet(\mathscr{B} \otimes \Lambda W)$$

and the injection is inducing the identity in cohomology after this identification. Thus

$$\psi = (m \circ \beta \circ i) : \mathscr{B} \otimes \Lambda W \xrightarrow{\simeq} \mathscr{C}$$

is a minimal Sullivan model for φ.

Given a second minimal Sullivan model $\psi' : \mathscr{B} \otimes \Lambda W' \xrightarrow{\simeq} \mathscr{C}$, apply Lemma 1.45 to the commutative square

$$\begin{tikzcd} \mathscr{B} \arrow[r] \arrow[d] & \mathscr{B} \otimes \Lambda W' \arrow[d, "\psi'", "\simeq"'] \\ \mathscr{B} \otimes \Lambda W \arrow[r, "\simeq", "\psi"'] & \mathscr{C} \end{tikzcd}$$

in order to extend $\mathrm{id}_{\mathscr{B}}$ to a morphism $\alpha : \mathscr{B} \otimes \Lambda W \xrightarrow{\cong} \mathscr{B} \otimes \Lambda W'$ with $\psi' \circ \alpha \simeq \psi \,\mathrm{rel}\, \mathscr{B}$. By Proposition 1.48 α is an isomorphism. This proves the uniqueness up to isomorphism claim. □

Corollary 1.50. *Let $\mathscr{A}$ be a connected CDGA. A minimal model $\psi_{\mathscr{A}} : \mathscr{M}_{\mathscr{A}} \xrightarrow{\simeq} \mathscr{A}$ exists and is unique up to CDGA isomorphism.*

Proof. Simply apply the previous theorem to the canonical injection $\varphi : k \hookrightarrow \mathscr{A}$. It asserts that there is a unique up to isomorphism minimal Sullivan model for φ, that we should call

$$\psi_{\mathscr{A}} : k \otimes \Lambda V \cong \Lambda V \xrightarrow{\simeq} \mathscr{A}.$$

Then $\psi_{\mathscr{A}} : \Lambda V \to \mathscr{A}$ is a minimal model for $\mathscr{A}$. □

1.5. **Rational formality.** Informally speaking, if one considers the differential forms on a connected manifold X, then the minimal model of $A_{dR}(X)$ is the smallest CDGA that encodes the so-called *de Rham homotopy type* of X. For instance, writing this minimal model as $\mathscr{M}_X = (\Lambda V, d)$, the generators V are exactly the indecomposables of $\mathscr{M}_X$, and if X is moreover simply connected, there is a spectacular duality theorem which states that $V^n \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_n(X), k)$ as k-vector spaces. Although we shall not prove this, it shall illustrate the type of data that a minimal model contains. It is moreover obvious, that if two manifolds X, Y have isomorphic minimal models $\mathscr{M}_X \cong \mathscr{M}_Y$, then they agree in cohomology.

The converse is not true. Generally, cohomology contains less information than the minimal model. A computation that illustrates this is carried out in an example in [FHT01, p. 157].

However, suppose we have a CDGA $(\mathscr{A}, d)$ such that there is a finite sequence of quasi-isomorphisms

$$(\mathscr{A}, d) \xleftarrow{\simeq} \mathscr{A}_0 \xrightarrow{\simeq} \mathscr{A}_1 \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} (H(\mathscr{A}),\, d = 0).$$

between $(\mathscr{A},\, d)$ and $(H(\mathscr{A}),\, d = 0)$. At an even stage i one has $\mathscr{A}_{i-1} \xleftarrow{\simeq} \mathscr{A}_i \xrightarrow{\simeq} \mathscr{A}_{i+1}$. Suppose they have minimal models $\mathscr{M}_{i-1}, \mathscr{M}_i$ and $\mathscr{M}_{i+1}$ respectively. Then there is the following diagram, where all the arrows are quasi-isomorphism.

$$\begin{tikzcd}
\mathscr{A}_{i-1} & & \mathscr{A}_i \arrow[ll, "\simeq"'] \arrow[rr, "\simeq"] & & \mathscr{A}_{i+1} \\
\mathscr{M}_{i-1} \arrow[u, "\psi_{i-1}"] & & \mathscr{M}_i \arrow[u, "\psi_i"'] \arrow[ull, "\simeq"] \arrow[urr, "\simeq"'] & & \mathscr{M}_{i+1}. \arrow[u, "\psi_{i+1}"']
\end{tikzcd}$$

It follows that $\mathscr{M}_i$ is a minimal model for $\mathscr{A}_{i-1}$ and $\mathscr{A}_{i+1}$ as well. By Theorem 1.32, all these minimal models are then isomorphic. Therefore, if such a sequence exists between $(\mathscr{A},\, d)$ and $(H(\mathscr{A}),\, d = 0)$, one recovers the minimal model of an algebra from its cohomology ring alone.

Definition 1.51. A connected CDGA $\mathscr{A}$ is *formal* (or *rationally formal*) if there exists a sequence of quasi-isomorphisms

$$(\mathscr{A}, d) \overset{\simeq}{\leftarrow} \mathscr{A}_0 \overset{\simeq}{\rightarrow} \mathscr{A}_1 \overset{\simeq}{\leftarrow} \dots \overset{\simeq}{\rightarrow} (H(\mathscr{A}),\, d = 0).$$

between $(\mathscr{A},\, d)$ and $(H(\mathscr{A}),\, d = 0)$. Such a sequence is called a *zig-zag*. Equivalently, $\mathscr{A}$ is formal if $\mathscr{M}_{\mathscr{A}}$ and $\mathscr{M}_{H(\mathscr{A})}$ are isomorphic.

A connected manifold X is said to be *formal* if its de Rham algebra $A_{dR}(X)$ is formal as a CDGA.

Example 1.52 (Spheres)**.** Recall that an n-dimensional sphere $\mathbb{S}^n$ has cohomology

$$H^\bullet(\mathbb{S}^n) \cong k[t]/(t^2), \text{ with } t \text{ of degree } n.$$

Now recall the inductive algorithm we have used in the Proposition 1.38. Choose an abstract vector x of degree n, which shall correspond to t in cohomology. Consider the free algebra $(\Lambda_n(x),\, d = 0)$.

Now, if n is odd, one has $x \cdot x = (-1)^n\; x \cdot x = 0$, and therefore $\Lambda_n(x) = k \;\oplus\; k \cdot x$. This is of course a Sullivan algebra and it is obviously minimal. Choose a differential form α that represents t in $H^\bullet(\mathbb{S}^n)$. Consider the morphism

$$\psi_n : \mathscr{M}_{\mathbb{S}^n} = \Lambda_n(x) \to A_{dR}(\mathbb{S}^n)$$

that sends x to α. It is a quasi-isomorphism, and thus it is the odd-dimmensional sphere's minimal model.

If n is even however, graded-commutativity does not imply that $x \cdot x = 0$ anymore. Hence in $\Lambda_n(x)$ there exist residual classes x^k for $k \geq 2$ that are non-zero. The next step of the algorithm is to add an element y of degree $2n - 1$ such that $dy = x \cdot x$. We thus obtain $(\Lambda_n(x) \otimes_d \Lambda_{2n-1}(y),\; dx = 0,\; dy = x^2)$. This is again a minimal algebra by construction. When taking cohomology, $[x]$ appears as a non-trivial class, however the powers $[x]^k$ become zero since x^2 is a coboundary. Again, if we choose a representative β of t in $H^\bullet(\mathbb{S}^n)$, then the morphism

$$\psi_n : \mathscr{M}_{\mathbb{S}^n} = \Lambda_n(x) \otimes_d \Lambda_{2n-1}(y) \to A_{dR}(\mathbb{S}^n)$$

that sends $x \mapsto \beta, y \mapsto 0$ is a quasi-isomorphism. This is the even-dimmensional sphere's minimal model.

In either case, $\mathbb{S}^n$ is rationally formal, since the morphism

$$\psi : \mathscr{M}_{\mathbb{S}^n} \to H^\bullet(\mathbb{S}^n)$$

which maps x to t (and y to 0 if n is even) is a quasi-isomorphism. Therefore there is an explicit zig-zag

$$A_{dR}(\mathbb{S}^n) \xleftarrow{\psi_n} \mathscr{M}_{\mathbb{S}^n} \xrightarrow{\psi} H^\bullet(\mathbb{S}^n)$$

which relies the de Rham CDGA of differential forms and its cohomology with trivial differential.

Example 1.53 (Complex projective spaces)**.** Recall that the cohomology of $\mathbb{CP}^n$ is

$$H^\bullet(\mathbb{CP}^n) = k[t]/(t^{n+1}), \text{ with } t \text{ of degree } 2.$$

It can be seen as a generalization of the computation we have done for $\mathbb{S}^2$, where starting from x^{n+1}, all higher powers of x become coboundaries. Indeed, one has

$$\mathscr{M}_{\mathbb{CP}^n} = (\Lambda_2(x) \otimes_d \Lambda_{2n+1}(y),\, dx = 0,\, dy = x^{n+1}).$$

The same argument used for rational formality of spheres applies in the case of complex projective spaces.

Example 1.54 (A non-formal CDGA)**.** Consider the free CDGA

$$\mathscr{M} = (\Lambda_1(x, y) \otimes_d \Lambda_1(z),\, dz = xy).$$

This is a minimal algebra by construction. Now suppose, for a proof by contradiction, that it is formal. Minimality then implies that $\mathscr{M}$ is the minimal model of its cohomology algebra $(H^\bullet(\mathscr{M}),\, d = 0)$ and therefore there should exist a direct quasi-isomorphism $\psi : \mathscr{M} \to H^\bullet(\mathscr{M})$.

The contradiction then comes from the fact that xz is a closed, but non-exact form in $\mathscr{M}$. Indeed, $d(xz) = dx \cdot z - x \cdot dz = -x^2y = 0$. It cannot be exact, since a degree 2 coboundary is coming from a degree 1 cocycle. But the only degree 1 elements with non-vanishing differential are scalar multiples of z, therefore a degree 2 coboundary is necessarily in the span of dz. On the other hand $\psi(xz) = \psi(x)\psi(z)$ is in the vector space $H^1(\mathscr{M}) \cdot H^1(\mathscr{M})$, which is null. This contradicts the injectivity of $H(\psi)$, therefore such a quasi-isomorphism cannot exist.

This happens to be the minimal model of the 3-dimensional Heisenberg nilmanifold $\mathrm{Nil} / \mathrm{Nil}_{\mathbb{Z}}$, and thus exhibits an example of a non-formal manifold by the same token.

Recall that a complex manifold M is said to be $\partial\bar\partial$ (or dd^c) if it satisfies the following property.

Lemma 1.55 ($\partial\bar\partial$-Lemma)**.** *If $\alpha \in A_{dR}(M, \mathbb{C})$ is a ∂- and $\bar\partial$-closed form that is moreover d- or ∂- or $\bar\partial$-exact, then $\alpha = \partial\bar\partial\beta$ for some β.*

Alternatively, one defines a differential $d^c = i(\bar\partial - \partial)$ which is a real operator

$$d^c : A_{dR}(M, \mathbb{R}) \to A_{dR}(M, \mathbb{R}) \subset A_{dR}(X, \mathbb{C})$$

and since over the complex forms

$$dd^c = 2i\partial\bar\partial = -d^c d$$

at first sight, the following statement is a refinement of the $\partial\bar\partial$-Lemma.

Lemma 1.56 (dd^c-Lemma)**.** *If $\alpha \in A_{dR}(M, \mathbb{R})$ is a d- and d^c-closed form that is also d- or d^c-exact, then $\alpha = dd^c\beta$ for some $\beta \in A_{dR}(M, \mathbb{R})$.*

Remark 1.57. In fact the two statements are equivalent.

Example 1.58. A classical result in Hodge theory states that every compact Kähler complex manifold is in fact dd^c. Complex projective spaces, and more generally smooth projective varieties over $\mathbb{C}$, can be seen as compact Kähler complex manifolds, and hence are dd^c.

However, not all dd^c manifolds are Kähler. The following theorem provides at once a plethora of examples of dd^c-manifolds, which are in general not Kähler.

Theorem 1.59. *Let $f : M^n \to N^n$ be a birational, holomorphic morphism between compact complex manifolds. If the dd^c-lemma holds for M^n, then it holds for N^n.*

We refer the intersted reader to [DGMS75, Theorem 5.22] for a sketch of the proof. We shall now prove the main theorem of this section, which is also due to [DGMS75].

Theorem 1.60. *A complex manifold M that is dd^c is rationally formal.*

Proof. Let A denote the de Rham complex $A_{dR}(M, \mathbb{R}) \subset A_{dR}(M, \mathbb{C})$ of real differential forms, $A^c = \operatorname{Ker} d^c$ be the sub-complex of d^c-closed forms and $H^c(M)$ be the quotient complex $A^c/d^c(A^c)$. One therefore has canonical maps

$$(A,\, d) \xleftarrow{\;i\;} (A^c,\, d) \xrightarrow{\;p\;} (H^c(M),\, d).$$

We claim that these are quasi-isomorphisms.

Let us first prove that the induced differential d on $H^c(M)$ is the null map. Suppose y is in A^c, then dy is a d-exact and d^c-closed form, therefore there is ω in A such that $dy = dd^c\omega = d^c(-d\omega)$. It follows that dy is d^c-exact and thus $[dy] = 0$ in $H^\bullet(H^c(M))$.

Consider now the morphism $H(i)$. It is injective, since if y is a d-closed form in A^c which is moreover d-exact in A, then the dd^c-Lemma provides a form ω in A such that $y = dd^c\omega$. As $d^c\omega$ is d^c-closed, it follows that $[y] = 0$ in $H^\bullet(A^c)$.

For surjectivity, let α be a certain cohomology class in $H^\bullet(A)$ and let $x \in A$ be a representative of α. It is not necessary for x to be d^c-closed. However, we claim that there is necessarily a form in the same cohomology class that is d^c-closed. Indeed, consider d^cx. It is d-closed and d^c-exact, therefore there is $\omega \in A$ such that $d^cx = dd^c\omega$. Then $x + d\omega$ is a form in A^c, since $d^cx + d^cd\omega = d^cx - dd^c\omega = 0$, that represents α.

Let us now prove that $H(p)$ is injective. If $y \in A^c$, $dy = 0$ is such that $[y] = 0$ in $A^c/d^c(A^c)$, then y is d-closed and d^c-exact. By the dd^c-Lemma, $y = dd^c\omega$ is also d-exact in A^c, therefore $[y] = 0$ in $H^\bullet(A^c)$.

For surjectivity of $H(p)$, start with a cohomology class α in $H^\bullet(H^c(M))$ which we have proven to be just $H^c(M)$, since the induced differential is null. Choose a representative y in A^c. It is not necessarily d-closed. We claim however that there is a d-closed representative in the same class $\alpha \in A^c/d^c(A^c)$. Consider dy, which is d^c-closed and d-exact, therefore $dy = dd^c\omega$. Then $y + d^c\omega$ is a d-closed form in A^c which represents α in $A^c/d^c(A^c)$. $\square$

This theorem establishes a bridge between rational homotopy theory and complex geometry. As an immediate consequence, we obtain formality statements of several classes of complex algebraic manifolds. Recall the following definition.

Definition 1.61. Let N be a lattice and set

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}.$$

A *rational polyhedral cone* in $N_\mathbb{R}$ is a cone of the form

$$\sigma = \mathrm{Cone}(v_1, \dots, v_k) = \left\{ \sum_{i=1}^{k} a_i v_i \,\middle|\, a_i \geq 0 \right\},$$

for some $v_1, \dots, v_k \in N$. It is called *strongly convex* if

$$\sigma \cap (-\sigma) = \{0\}.$$

A *fan* in $N_\mathbb{R}$ is a finite collection Σ of strongly convex rational polyhedral cones such that

(1) every face of a cone in Σ is also a cone in Σ;
(2) for any $\sigma, \tau \in \Sigma$, the intersection

$$\sigma \cap \tau$$

is a face of both σ and τ.

Now, a fundamental result in toric geometry is the one-to-one correspondence between normal toric varieties with acting torus T and fans in the so-called *cocharacter lattice* $X_*(T)$ (see e.g. [CLS11, Theorem 3.1.5 & Corollary 3.1.8]).

Corollary 1.62. *A complete smooth toric variety X_Σ is rationally formal.*

Remark 1.63. Starting with dimension 3, there are examples of complete smooth torics which are neither projective, nor Kähler. The corollary thus exhibits examples of non-Kähler formal manifolds.

Proof of the Corollary. It suffices to prove that X_Σ is dd^c. This is a direct corollary of Theorem 1.59, of the Toric Chow Lemma and of the resolution of singularities for toric varieties. A reference for the latter is [CLS11, Theorem 6.1.18]. Indeed, a complete fan Σ has a refinement Σ' such that $X_{\Sigma'}$ is projective smooth. Moreover, a refinement corresponds to a birational toric morphism

$$f : X_{\Sigma'} \to X_\Sigma$$

which can be thought of as a birational holomorphic mapping between two compact complex manifolds. Since $X_{\Sigma'}$ is known to be dd^c, by Theorem 1.59 X_Σ is dd^c as well. □

Now, recall that in horospherical geometry there is an analogous construction of fans, called *coloured fans*. The same argument applied in the horospherical setting yields a more general result.

Corollary 1.64. *A complete smooth horospherical variety X_{Σ^c} is rationally formal.*

Proof. Consider the coloured fan Σ^c. Now, [Mon26, Corollary 7.16] asserts that the horospherical variety X_{Σ^c} is projective if and only if the toric variety X_Σ is. Starting with the coloured fan Σ^c, one applies the argument used in 1.62 to the underlying colourless fan Σ and obtains a refinement Σ' such that $X_{\Sigma'}$ is toric projective smooth. It suffices now to re-attach the colours to Σ' by setting

$$\mathcal{F}(\sigma'^c) := \{\alpha \in \mathscr{C} : u_\alpha \in \sigma'\}.$$

Again, the refinement morphism

$$f^c : X_{\Sigma'^c} \to X_{\Sigma^c}$$

can be thought of as a birational holomorphic mapping between two compact complex manifolds, and $X_{\Sigma'^c}$ being smooth projective is *a fortiori* dd^c. This proves the claim. □

This bridge between rational homotopy theory and complex geometry can also be crossed the other way around. For instance, it is often a difficult question to determine whether a given complex manifold is Kähler or not. Non-formality then rises as a sufficient condition for non-Kählerness. This strategy is used in [CFG86] and [TO97] to exhibit explicit examples of compact symplectic manifolds which are not Kähler.

2. The equivariant setting

This section provides an outline of the topological equivariant cohomology theory. A standard reference of the subject is [GS99]. A more concise overview of the theory is given in [Mei06].

2.1. **Borel's construction. Equivariant cohomology.**

Let G be a compact Lie group acting smoothly on a smooth manifold X. Naturally, one is led to study the singular cohomology $H^\bullet(X/G, A)$ over a ring A of one's choice. However, it so happens that when G does not act freely on X, the aforementioned cohomology ring may behave poorly. For instance the quotient X/G may present singularities or may not be a manifold at all. In this context, Borel's fibration replaces the quotient X/G by a better behaved topological space X_G, which is homotopy equivalent to X/G if the action of G is free. In other words, one considers the more general cohomology ring $H^\bullet(X_G, A)$ which is always defined and is canonically isomorphic to $H^\bullet(X/G, A)$ whenever the G-action on X is free.

Let us proceed to Borel's construction of X_G.

Definition 2.1. Let G be a topological group. A *principal G-bundle* is a map $\pi : P \to B$ together with a free continuous G-action on P such that

(1) The fibers are G-orbits, meaning
$$\pi^{-1}(b) = G \cdot p$$
for every $p \in \pi^{-1}(b)$.

(2) The bundle is G-equivariantly locally trivial, meaning that every $b \in B$ has a neighborhood U such that $\pi^{-1}(U) \cong U \times G$ in a G-equivariant way, where G acts on $U \times G$ by
$$h \cdot (x, g) = (x, hg).$$

Definition 2.2. Let G be a compact Lie group. A *classifying bundle* for G is a principal G-bundle $EG \to BG$ with a CW-complex base BG, satisfying the following universal property. For any principal G-bundle $E \to B$ with a CW-complex base B, there is a unique up to G-G-equivariant homotopy map $f : B \to BG$ such that E is isomorphic to the pullback f^*EG.

$$\begin{array}{ccc} E & & EG \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\exists f} & BG. \end{array}$$

The base BG of a classifying bundle is called a *classifying space* and the map f is called a *classifying map* of the principal bundle $E \to B$.

The following proposition provides explicit examples of classifying bundles for G.

Proposition 2.3. [Ben98, Theorem 2.4.8] *A principal G-bundle $E \to B$ with a contractible total space E and with a CW-complex base B is a classifying bundle for G.*

Example 2.4. Consider a compact torus $T = (\mathbb{S}^1)^r$. Recall the definitions of the following CW-complexes

$$\mathbb{S}^\infty = \Big\{(x_0, x_1, \dots) \in \mathbb{R}^\infty \mid x_i = 0 \text{ for all but finitely many } i, \ \sum_i x_i^2 = 1\Big\},$$

$$\mathbb{CP}^\infty = \Big\{(z_0, z_1, \dots) \in \mathbb{C}^\infty \setminus \{0\} \mid z_i = 0 \text{ for all but finitely many } i\Big\} \Big/ \sim,$$

where $(z_0, z_1, \dots) \sim (\lambda z_0, \lambda z_1, \dots)$ for any $\lambda \in \mathbb{C}^\times$. A principal T-bundle is given by

$$ET = (\mathbb{C}^\infty \setminus \{0\})^r \to BT = (\mathbb{CP}^\infty)^r$$

with a T-action defined by

$$(t_1, \dots, t_r) \cdot (v_1, \dots, v_r) = (t_1 v_1, \dots t_r v_r).$$

The total space ET is homotopy equivalent to $(\mathbb{S}^\infty)^r$. The latter is weakly contractible, meaning $\pi_n(\mathbb{S}^\infty, \mathrm{pt}) = 0$ for $n > 0$, is a CW-complex, so by Whitehead's theorem it is contractible. Since the base is a CW-complex, this is a classifying bundle for T.

Definition 2.5. Given a compact Lie group G and a G-action on a smooth manifold X, choose a classifying bundle $EG \to BG$. Consider the diagonal action of G on $X \times EG$. The *Borel fibration associated to* X is a principal G-fibration

$$X_G := (X \times EG)/G \to EG/G = BG.$$

Definition 2.6. The G-equivariant cohomology of X is the singular cohomology of the aforementioned X_G, namely

$$H^\bullet_G(X, A) = H^\bullet(X_G, A).$$

Proposition 2.7. *The G-equivariant cohomology of X does not depend on the choice of the classifying space EG in Borel's construction.*

Proof. Suppose that

$$EG \to BG \qquad \text{and} \qquad (EG)' \to (BG)'$$

are two universal principal G-bundles. By the universal property of $(EG)' \to (BG)'$, there exists a map

$$f : BG \to (BG)'$$

such that

$$EG \cong f^*(EG)'.$$

Equivalently, there is a G-equivariant map

$$\widetilde{f} : EG \to (EG)'$$

covering f, unique up to G-equivariant homotopy. Similarly, there is a map

$$g : (BG)' \to BG$$

with a lift

$$\widetilde{g} : (EG)' \to EG,$$

unique up to G-equivariant homotopy.

By the unicity statement of the universal property for classifying bundles, the map $g \circ f$ is G-equivariantly homotopic to id_{BG}, that is $g \circ f \simeq_G \mathrm{id}_{BG}$, and the map $f \circ g \simeq_G \mathrm{id}_{(BG)'}$.

The composites $\widetilde{g} \circ \widetilde{f}$ and $\widetilde{f} \circ \widetilde{g}$ are G-equivariantly homotopic to the respective identity maps by the homotopy lifting property for principal G-bundles. Hence the induced maps

$$(X \times EG)/G \longrightarrow (X \times (EG)')/G \qquad \text{and} \qquad (X \times (EG)')/G \longrightarrow (X \times EG)/G$$

are homotopy inverses. Therefore

$$H^\bullet((X \times EG)/G, A) \cong H^\bullet((X \times (EG)')/G, A).$$

□

It follows that H_G is a contravariant functor from the category of smooth manifolds endowed with a G-action to the category of A-modules. In particular, the G-equivariant map $X \to \mathrm{pt}$ induces an A-module morphism

$$H_G^\bullet(\mathrm{pt}) = H^\bullet(BG) \to H_G^\bullet(X)$$

and thereby endows $H_G^\bullet(X)$ with a natural $H^\bullet(BG)$-module structure. See example 3.16 for an explicit computation.

2.2. **The Cartan model.** The Borel construction provides a topological definition of equivariant cohomology. For computational purposes, however, it is often more convenient to work with a CDGA model. When G is a compact connected real Lie group acting smoothly (or by biholomorphisms) on a differential (or complex) manifold X, such a model is provided by the Cartan model.

Let $\mathfrak{g}$ denote the Lie algebra of G. Since G is compact, $\mathfrak{g}$ is a finite-dimensional vector space over k. Consider the symmetric algebra $S(\mathfrak{g}^\vee)$. One can think of this the following way. Fix a basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$ and consider its dual basis $\xi^1, \dots, \xi^n$ in $\mathfrak{g}^\vee$. The algebra $S(\mathfrak{g}^\vee)$ is then the polynomial algebra in the indeterminates $\xi^1, \dots, \xi^n$, which are decreed to be of degree 2 (see formula $(*)$ for the reason of this choice of degree).

Recall that there is a canonical action of G on $\mathfrak{g}^\vee$, namely the coadjoint action, which extends canonically to $S(\mathfrak{g}^\vee)$. It is defined by

$$(g \cdot \alpha)(\xi) = \alpha(\mathrm{Ad}_{g^{-1}} \xi) = \frac{d}{dt}\bigg|_{t=0} g^{-1} \exp(t\xi) g, \quad \alpha \in \mathfrak{g}^\vee,\ \xi \in \mathfrak{g}.$$

Definition 2.8. Let $A^\bullet(X)$ denote the CDGA of smooth differential forms on X. The Cartan complex is defined as

$$C_G^\bullet(X) = (S^\bullet(\mathfrak{g}^\vee) \otimes A^\bullet(X))^G,$$

where G acts diagonally on $S(\mathfrak{g}^\vee) \otimes A^\bullet(X)$. For $\omega \in C_G(X)$, its differential is given by

$$(*) \qquad d_G(1 \otimes \omega) = 1 \otimes d\omega - \sum_{i=1}^n \xi^i \otimes \iota_{\xi_i} \omega$$

where ι_{ξ_i} denotes contraction with the vector field induced by $\xi_i \in \mathfrak{g}$. The grading on $C_G^\bullet(X)$ is defined to be the total grading of the bicomplex $(S^\bullet(\mathfrak{g}^\vee) \otimes A^\bullet(X))^G$, namely

$$C_G^k(X) = \bigoplus_{2p+q=k} (S^p(\mathfrak{g}^\vee) \otimes A^q(X))^G,$$

where $S^p(\mathfrak{g}^\vee)$ are homogeneous polynomials in $\xi^1, \dots, \xi^n$ of degree p.

Equivalently, if one wants to avoid choosing a basis for $\mathfrak{g}$, the Cartan complex can be identified with the space of G-equivariant polynomial maps

$$\alpha : \mathfrak{g} \to A^\bullet(X).$$

In this description the differential is given by

$$(d_G\alpha)(\xi) = d\alpha(\xi) - \iota_{\xi_X}(\alpha(\xi)), \quad \xi \in \mathfrak{g},$$

where ξ_X denotes the vector field on X induced by ξ and ι_{ξ_X} denotes contraction with this vector field. The Cartan differential squares to zero on the G-invariant subcomplex by Cartan's magic formula, so $C_G(X)$ is indeed a CDGA.

The following fundamental result is due to Cartan [Car50].

Theorem 2.9 (Equivariant de Rham theorem)**.** *Let G and X be as above. The Cartan model $C_G(X)$ is a CDGA model for the equivariant cohomology of X relative to G, namely*

$$H_G^\bullet(X) \cong H^\bullet(C_G(X)).$$

Remark 2.10. Notice that when G is abelian, the coadjoint action of G on $\mathfrak{g}^\vee$ is trivial, and then so is the action on $S(\mathfrak{g}^\vee)$. Indeed, for ξ in $\mathfrak{g}$

$$\mathrm{Ad}_g\,\xi = \frac{d}{dt}\bigg|_{t=0}(g\exp(t\xi)g^{-1}) = \frac{d}{dt}\bigg|_{t=0}\exp(t\xi) = \xi.$$

Therefore if G is an abelian compact connected Lie group acting smoothly on X, e.g. if $G = T$ is a torus with Lie algebra $\mathfrak{t}$, then the Cartan model becomes

$$C_T(X) = S(\mathfrak{t}^\vee) \otimes A^\bullet(X)^T$$

since every element in $S(\mathfrak{t}^\vee)$ is T-equivariant.

2.3. **The spectral sequence of the Cartan model. Equivariant formality.** When $A = k$ is a field, one of the main techniques to compute equivariant cohomology is the Cartan model spectral sequence. We cannot resist quoting Vakil [Vak25] on this matter :

> *It has been suggested that the name 'spectral' was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.*

Recall that one can associate to a double complex

$$C = \bigoplus_{p,q\in\mathbb{Z}} C^{p,q}$$

with coboundary operators

$$d : C^{p,q} \to C^{p,q+1}, \quad \delta : C^{p,q} \to C^{p+1,q}$$

a spectral sequence $(E_r^{\bullet,\bullet}, d_r)_{r\geq 0}$ which is defined inductively on the $r+1$ page by

$$E_{r+1}^{p,q} = \frac{\mathrm{Ker}\, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}}{\mathrm{Im}\, d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}}$$

and d_r being the map induced by the differential $d+\delta$ on the r page.

The spectral sequence is said to degenerate (or collapse) if there is $n \in \mathbb{N}$ such that

$$E_n^{\bullet,\bullet} = E_{n+1}^{\bullet,\bullet} = E_{n+2}^{\bullet,\bullet} = \ldots =: E_\infty^{\bullet,\bullet}.$$

If this is the smallest such n, then we say that $(E_r^{\bullet,\bullet}, d_r)_{r\geq 0}$ degenerates at page n.

On the other hand, convergence of spectral sequences is a slightly different matter. Recall that to a double complex $C^{\bullet,\bullet}$ one can associate a single graded *total complex*

$$(\operatorname{tot} C)^k = \bigoplus_{p+q=k} C^{p,q}$$

on which one can naturally define two decreasing filtrations. Start by setting

$$(F_I^p \operatorname{tot}(C))^q = \bigoplus_{r \geq p} C^{r,q-r} \text{ and}$$
$$(F_{II}^p \operatorname{tot}(C))^q = \bigoplus_{r \geq p} C^{q-r,r}.$$

The filtrations on $\operatorname{tot}(C)$ are defined to be

$$F_I^p \operatorname{tot}(C) = \bigoplus_{q \in \mathbb{Z}} (F_I^p \operatorname{tot}(C))^q \qquad \text{(the column-wise filtration)}$$
$$F_{II}^p \operatorname{tot}(C) = \bigoplus_{q \in \mathbb{Z}} (F_{II}^p \operatorname{tot}(C))^q \qquad \text{(the row-wise filtration).}$$

Note that the total differential respects these filtrations, i.e.

$$(d+\delta)(F_*^p \operatorname{tot}(C)) \subset F_*^p \operatorname{tot}(C).$$

The column-wise filtration will be the better suited one for the purposes of this section. We shall simply write F^pC^n for $(F_I^p \operatorname{tot}(C))^n$ and F^pC for $F_I^p \operatorname{tot}(C)$. Since $d+\delta$ respects the filtration, the cohomology of the total complex $H(C) = H^\bullet(\operatorname{tot}(C), d+\delta)$ also inherits a filtration

$$F^pH^n(C) = \operatorname{Im}\left(H^n(F^pC, d+\delta) \xrightarrow{H^n(\text{inclusion})} H^n(C)\right),$$

where again, the filtration on $H(C)$ is obtained by summing $F^pH^n(C)$ over all n.

Definition 2.11. The spectral sequence $(E_r^{\bullet,\bullet}, d_r)$ is said to *converge* if for any bidegree (p,q) there is $N = N(p,q)$ such that for all $n \geq N$ one has

$$E_n^{p,q} \cong F^pH^{p+q}(C)/F^{p+1}H^{p+q}(C) =: \operatorname{gr}^p H^{p+q}(C).$$

One may also write $E_\infty^{p,q} \cong \operatorname{gr}^p H^{p+q}(C)$ or

$$E_r^{p,q} \implies H^{p+q}(C)$$

for short. In the last notation, r in $E_r^{p,q}$ suggests that the computation is done starting from the rth page.

Suppose now that this filtration is *bounded*, meaning that for each n in $\mathbb{Z}$ there is $s = s(n)$ and $t = t(n)$ so that

$$\{0\} \subset F^sC^n \subset F^{s-1}C^n \subset \ldots \subset F^{t+1}C^n \subset F^tC^n = \operatorname{tot}(C)^n.$$

Notice that this is always the case for a first quadrant double complex. Under this assumption, the spectral sequence converges

$$E_\infty^{p,q} \cong F^pH^{p+q}(C)/F^{p+1}H^{p+q}(C).$$

We direct the interested reader to [McC01, Theorem 2.6] for further details.

Consider now the Cartan model $C_G(X)$ as a double complex, so that its CDGA grading stays consistent with the total degree of the double complex, namely

$$E_0^{p,q} = C_G^{p,q} := (S^p(\mathfrak{g}^\vee) \otimes A^{q-p}(X))^G$$

and the differentials of the double complex, with the notations above, being given by

$$d = 1 \otimes d_{A(X)}, \qquad \delta = -\sum_{i=1}^{n} \xi^i \otimes \iota_{\xi_i}.$$

Notice that $d + \delta = d_G$. The spectral sequence thus defined is the *Cartan model spectral sequence.* It is first quadrant by construction, therefore it is necessarily convergent.

Theorem 2.12. *The Cartan model spectral sequence converges*

$$E_\infty^{p,q} \cong \mathrm{gr}^p\, H^{p+q}(C_G(X)) \cong \mathrm{gr}^p\, H_G^{p+q}(X).$$

Let us now compute its first page.

Lemma 2.13. *The first page of the Cartan model spectral sequence reads*

$$E_1^{p,q} \cong S^p(\mathfrak{g}^\vee)^G \otimes H^{q-p}(A(X)).$$

Proof. On the 0-page, one has a differential

$$d_0 : C_G^{p,q} \to C_G^{p,q+1}$$

induced by d_G. This corresponds to

$$1 \otimes d_{A(X)} = (S^p(\mathfrak{g}^\vee) \otimes A^{q-p}(X))^G \to (S^p(\mathfrak{g}^\vee) \otimes A^{q+1-p}(X))^G.$$

Therefore, the E_1 term in the spectral sequence is

$$E_1^{p,q} = \big(S^p(\mathfrak{g}^\vee) \otimes H^{q-p}(A(X))\big)^G.$$

Recall that, since G is connected, G acts trivially on $H(A(X))$. The reason for this is that, for any g in G there is a path between e and g, since G is Lie connected, hence path connected. But that means that there is a homotopy between id_X and $g_X : X \to X$. It follows by homotopy invariance of the cohomology that for any closed form $\omega \in A(X)$ one has $[g_X^*\omega] = [\omega]$. This simplifies the formula for the first page further, namely

$$E_1^{p,q} = S^p(\mathfrak{g}^\vee)^G \otimes H^{q-p}(A(X)).$$

□

This is the algebraic analogue of the Leray-Serre spectral sequence associated to the Borel fibration $X \hookrightarrow X_G \to BG$. When BG is simply connected, which is the case whenever G is compact connected, the second page of Leray-Serre reads

$$E_2^{p,q} = H^p(BG, k) \otimes H^q(X, k) \implies H_G^{p+q}(X)$$

and converges to the equivariant cohomology of X. In fact, the first page of the Cartan spectral sequence is isomorphic to the second page of the Leray-Serre spectral sequence up to bidegree shifts. Notably, there is a CDGA isomorphism

$$\Phi : S^\bullet(\mathfrak{g}^\vee)^G \xrightarrow{\cong} H^{2\bullet}(BG, k)$$

given by the Chern-Weil homomorphism, see e.g. [Dup78, Theorem 8.1] for $k = \mathbb{R}$.

Remark 2.14. This is consistent with the Equivariant de Rham Theorem and with Theorem 2.12, since for $X = \mathrm{pt}$ the Cartan model spectral sequence must converge

$$E_0^{p,p} = S^p(\mathfrak{g}^\vee) \implies H_G^{2p}(\mathrm{pt}, k) = H^{2p}(BG, k).$$

Recall that $H_G(X, k)$ has a canonical $H(BG, k)$-module structure given by the pull-back of the Borel fibration. Under the identifications above, it amounts to say that $H_G(X, k)$ is a $S(\mathfrak{g}^\vee)^G$-module. Moreover, the formula for the E_1 page of the Cartan model spectral sequence yields the following.

Theorem 2.15. [GS99, Theorem 6.6.1] *If* $\dim H(X,k)$ *is finite, then* $H_G(X,k)$ *is a finite type* $S(\mathfrak{g}^\vee)^G$*-module.*

Proof. Clearly $E_1 = S(\mathfrak{g}^\vee)^G \otimes H(X,k)$ is a free $S(\mathfrak{g}^\vee)^G$-module, since $H(X,k)$ is a k-vector space. Thus, if $\dim H(X,k)$ is supposed finite, E_1 is consequently a finite type $S(\mathfrak{g}^\vee)^G$-module. Moreover, under our assumptions of G being compact connected, $S(\mathfrak{g}^\vee)^G$ is known to be noetherian by a theorem of Chevalley [Che55]. Therefore any submodule of E_1 is again of finite type. This is the case for $\operatorname{Ker} d_1$. Quotients of finite type modules are trivially of finite type, therefore one obtains by an immediate induction that E_r is of finite type for all $r \geq 1$.

Now, the fact that $\dim H(X,k)$ is finite forces the spectral sequence to degenerate. Indeed, the $(E_1^{i,j})_{(i,j)}$ page has non-zero contributions only for $0 \leq j \leq \dim H(X,k)$. Passing to the next pages can only make disappear or modify already existing contributions. In other words, outside of the row range $0 \leq j \leq \dim H(X,k)$, any $E_r^{i,j}$ is equal to zero. But clearly for r big enough, either the domain or the target of

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

is outside of this range. Consider for example $r \geq \dim H(X,k) + 2$. It follows that d_r, and all the subsequent differentials for that matter, are null. Thus $E_\infty = E_{\dim H(X,k)+2}$ is a genuine page and therefore is of finite type over $S(\mathfrak{g}^\vee)^G$.

Lastly,

$$E_\infty \cong \operatorname{gr} H_G(X,k) = \bigoplus_{p,q} \operatorname{gr}^p H_G^{p+q}(X,k).$$

This implies that $H_G(X,k)$ is of finite type. To show this, choose $S(\mathfrak{g}^\vee)^G$-generators $\hat{h}_i$ of $\operatorname{gr} H_G(X)$ for some finite index set I. Next, for every i, lift $\hat{h}_i \in \operatorname{gr}^{p_i} H_G(X)$ to $h_i \in F^{p_i} H_G(X)$. Now, for any given $h \in H_G(X)$, there exists p such that h is in $F^p H_G(X) \setminus F^{p+1} H_G(X)$. Clearly, there also exist $a_i \in S(\mathfrak{g}^\vee)^G$ such that $\bar{h} = \sum_i a_i \bar{h}_i$ when projected in $\operatorname{gr}^p H_G(X)$. Denote by $J(p) \subset I$ the index subset such that $\bar{h}_j \neq 0$ in $\operatorname{gr}^p H_G(X)$ whenever $j \in J$. It follows that

$$h - \sum_{j \in J} a_j h_j \in F^{p+1} H_G(X).$$

Since the filtration is bounded in each fixed total degree, this process can be iterated only finitely many times. Hence h can be written as a finite $S(\mathfrak{g}^\vee)^G$-linear combination of the h_i. Therefore $H_G(X,k)$ is of finite type. ◻

Definition 2.16. The manifold X is called *equivariantly formal* if the Cartan model spectral sequence degenerates at page 1.

Remark 2.17. In spite of the similar terminology, rational formality and equivariant formality are two independent properties. Generally, neither implies the other, even in the most trivial cases, like for $G = \{1\}$.

Theorem 2.18. [GS99, Theorem 6.6.2] *If* X *is equivariantly formal, then* $H_G(X,k)$ *is a free* $S(\mathfrak{g}^\vee)^G$*-module.*

Proof. By definition $E_\infty = E_1 = S(\mathfrak{g}^\vee)^G \otimes H(X,k)$ is free, since $H(X,k)$ is a k-vector space. On the other hand, $E_\infty \cong \operatorname{gr} H_G(X,k) = \bigoplus_{p,q} \operatorname{gr}^p H_G^{p+q}(X,k)$ as $S(\mathfrak{g}^\vee)^G$-modules. This implies that $H_G(X,k)$ is free over $S(\mathfrak{g}^\vee)^G$, by an induction argument similar to the one used in Theorem 2.15. ◻

A sufficient condition for equivariant formality is the following.

Proposition 2.19. *If X is such that $H^n(X,k)=0$ for n odd, then X is equivariantly formal.*

Proof. Consider the differential of the first page

$$d_1 : S^p(\mathfrak{g}^\vee)^G \otimes H^{q-p}(X,k) \to S^{p+1}(\mathfrak{g}^\vee)^G \otimes H^{q-(p+1)}(X,k).$$

Notice that either $q-p$ or $q-(p+1)$ is odd, therefore $d_1=0$.

Assuming all the differentials $d_1,\dots,d_{r-1}$ are null, by induction $E_1=E_2=\dots=E_r$, and since the r-stage differential maps $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$, one has

$$d_r : S^p(\mathfrak{g}^\vee)^G \otimes H^{q-p}(X,k) \to S^{p+r}(\mathfrak{g}^\vee)^G \otimes H^{q-r-(p+r)+1}(X,k).$$

Again, either $q-p$ or $q-r-(p+r)+1=q-p+1-2r$ is odd, whence $d_r=0$. This concludes the argument. $\square$

Example 2.20. Projective spaces are among the most simple equivariantly formal manifolds (provided a G-action is set). Indeed, for any $n\in\mathbb{N}$ one has

$$H^\bullet(\mathbb{CP}^n,k)\cong\frac{k[z]}{(z^{n+1})},\qquad \text{with } |z|=2.$$

Therefore $H^{\mathrm{odd}}(\mathbb{CP}^n,k)=0$.

Corollary 2.21. *A complete smooth G-spherical variety X is equivariantly formal. In particular, complete smooth horospherical varieties and complete smooth toric varieties are equivariantly formal.*

Proof. The choice of the compact connected Lie subgroup $K\subset G$ used for the Borel construction X_K is irrelevant here. Indeed, it suffices to show that $H^n(X,k)=0$ for n odd. That this is the case is a consequence of the theory of Chow groups on spherical varieties. Since X is a complete smooth spherical variety, one has an isomorphism

$$A_\bullet(X)\cong H_\bullet(X,\mathbb{Z})$$

between the algebraic cycles on X and its integral cohomology [Per12, Corollary 2.2.4]. Since a cycle $[Z]$ appears in degree double to the dimension of the subvariety Z, it follows that $H_n(X,\mathbb{Z})=0$ for n odd. By the Universal Coefficient Theorem and Poincaré duality, one concludes that

$$H^n(X,k)=0 \text{ for } n \text{ odd}$$

and $k=\mathbb{Q},\mathbb{R}$ or $\mathbb{C}$. $\square$

Remark 2.22. Both the completeness and smoothness are required. See, for instance, [CLS11, Example 12.3.8] for a complete toric variety with non-trivial odd cohomology.

On the other hand, there exist very simple examples of smooth spherical varieties which have odd cohomology. Think, for instance, of $\mathbb{C}^\times$ as a toric variety with a T-action given by multiplication. It is clear that $H^1(\mathbb{C},k)\cong k$, since $\mathbb{C}^\times\simeq\mathbb{S}^1$ are homotopy equivalent.

3. Pluripotential homotopy theory

3.1. **Preliminaries.** In a nutshell, rational homotopy theory is the study of the category $\mathbf{CDGA}[\text{quiso}^{-1}]$. One of its main applications is the study of smooth differential forms on smooth manifolds. Complex manifolds, on the other hand, give rise to differential graded algebras with additional structure. In particular, if X is a complex manifold, its complex structure induces a canonical bigrading

$$A^{\bullet}_{dR}(X, \mathbb{C}) = \bigoplus_{p,q} A^{p,q}(X)$$

where $A^{p,q}(X)$ is the space of forms with p-holomorphic and q-antiholomorphic primitive components. This motivates the following definition.

Definition 3.1. Recall that a bicomplex over $\mathbb{C}$ consists of a bigraded $\mathbb{C}$-vector space $A^{\bullet,\bullet}$ equipped with two $\mathbb{C}$-linear maps

$$\partial : A^{p,q} \longrightarrow A^{p+1,q}, \qquad \bar{\partial} : A^{p,q} \longrightarrow A^{p,q+1},$$

called the *differentials*, satisfying

$$\partial^2 = 0, \qquad \bar{\partial}^2 = 0, \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

A *commutative bidifferential bigraded algebra*, or CBBA for short, is a commutative algebra object in the category of bicomplexes. It consists of a bicomplex A and of a map of bicomplexes $A \times A \to A$ such that

$$A^{p,q} \cdot A^{r,s} \subset A^{p+r,q+s},$$

as well as a unit $1 \in A^{0,0}$ satisfying associativity, unitality, graded-commutativity and the differentials are graded derivations on A. That is, with respect to the total degree

$$|x| = p + q, \qquad x \in A^{p,q},$$

the algebra is graded-commutative

$$xy = (-1)^{|x||y|} yx,$$

and both ∂ and $\bar{\partial}$ are graded derivations

$$\partial(xy) = \partial x \cdot y + (-1)^{|x|} x \cdot \partial y,$$
$$\bar{\partial}(xy) = \bar{\partial} x \cdot y + (-1)^{|x|} x \cdot \bar{\partial} y.$$

Remark 3.2. Given a CBBA $(A^{\bullet,\bullet}, \partial, \bar{\partial})$ one recovers a CDGA $(A^{\bullet+\bullet}, \partial + \bar{\partial})$.

Definition 3.3. Let A and B be two bicomplexes. A *morphism of bicomplexes* is a linear map of the underlying vector spaces $f : A \to B$ that

(i) preserves the bigrading, that is $f(A^{p,q}) \subset B^{p,q}$ for any natural p and q;
(ii) commutes with the differentials, namely

$$f \circ \partial_A = \partial_B \circ f, \qquad f \circ \bar{\partial}_A = \bar{\partial}_B \circ f.$$

A *CBBA morphism* is a morphism of bicomplexes compatible with the multiplication and the unit.

For a bicomplex $(A^{\bullet,\bullet}, \partial, \bar{\partial})$ there are several cohomologies to be considered. All the coefficients are complex. Firstly, there is the usual de Rham cohomology, defined as

$$H_{dR}(A) = \frac{\operatorname{Ker}(\partial + \bar{\partial})}{\operatorname{Im}(\partial + \bar{\partial})}.$$

Next, one can consider

$$H_\partial(A) = \frac{\operatorname{Ker}\partial}{\operatorname{Im}\partial}, \qquad H_{\bar\partial}(A) = \frac{\operatorname{Ker}\bar\partial}{\operatorname{Im}\bar\partial}$$

called the *Dolbeault* and the *conjugate Dolbeault* cohomologies respectively.

Lastly, there are the *Aeppli* and *Bott-Chern* cohomologies

$$H_A(A) = \frac{\operatorname{Ker}\partial\bar\partial}{\operatorname{Im}\partial + \operatorname{Im}\bar\partial}, \qquad H_{BC}(A) = \frac{\operatorname{Ker}\partial \cap \operatorname{Ker}\bar\partial}{\operatorname{Im}\partial\bar\partial}.$$

It is a small exercice to prove that the bigrading on $A^{\bullet,\bullet}$ induces well-defined bigradings on all the aforementioned cohomologies, except de Rham. That is, for any cohomology class $\alpha \in H^\bullet_*(A)$ with $* \in \{\partial, \bar\partial, A, BC\}$, if $a \in A^{p,q}$ and $a' \in A^{r,s}$ are pure type representatives for the respective cohomology theory $*$, then $p = r$ and $q = s$. These are therefore functors from bicomplexes to bigraded (or just graded for H_{dR}) vector spaces and we shall write $H_*(f)$ for the induced maps.

The following definition is due to Stelzig [Ste25].

Definition 3.4. A bicomplex morphism $f : A \to B$ is called a *pluripotential quasi-isomorphism* if it induces isomorphisms $H_A(f)$ and $H_{BC}(f)$ in Aeppli and Bott-Chern. If $f : A \to B$ is a CBBA morphism, it is a *pluripotential quasi-isomorphism* if its associated bicomplex morphism is one.

One of the motivations of this definition is the following result.

Theorem 3.5. [Ste25, Theorem C] *If $f : A \to B$ is a pluripotential quasi-isomorphism bewteen bicomplexes, then $H_{dR}(f), H_\partial(f)$ and $H_{\bar\partial}(f)$ are isomorphisms. Conversely, if A and B have bounded antidiagonals, assuming $H_\partial(f)$ and $H_{\bar\partial}(f)$ are isomorphisms, f is a pluripotential quasi-isomorphism.*

Remark 3.6. In fact, Stelzig's theorem is a stronger statement. In addition to the implications mentioned in Theorem 3.5, it asserts that a map of bicomplexes $f : A \to B$ is an isomorphism in the homotopy category **HoBiCo** if and only if it is a pluripotential quasi-isomorphism. In other words, pluripotential quasi-isomorphisms are the *correct* notion of quasi-isomorphisms to consider in the category of bicomplexes.

Similarly to the case of CDGAs, one is led to study the category **CBBA**[pluriquiso^{-1}]. In light of this, the following definition arises naturally.

Definition 3.7. A CBBA A is called *strongly formal* if it is isomorphic to a CBBA $(H, \partial = 0, \bar\partial = 0)$ in this homotopy category, i.e. if there exists a zig-zag of pluripotential quasi-isomorphisms

$$A \leftarrow A_0 \rightarrow A_1 \leftarrow \ldots \rightarrow (H, \partial = 0, \bar\partial = 0).$$

A compact complex manifold X is *strongly formal* if $A^{\bullet,\bullet}_{dR}(X)$ is a strongly formal CBBA.

Lastly, recall that for a bicomplex A there is a diagram

$$\begin{tikzcd} & H_{BC}(A) \arrow[dl] \arrow[d, "i"] \arrow[dr] & \\ H_\partial(A) \arrow[dr] & H_{dR}(A) \arrow[d] & H_{\bar\partial}(A) \arrow[dl] \\ & H_A(A) & \end{tikzcd}$$

induced by $\mathrm{id} : A \to A$. Now, if i is injective, then a d-exact element x in $\operatorname{Ker}\partial \cap \operatorname{Ker}\bar\partial$ is necessarily $\partial\bar\partial$-exact, which amounts to say that A satisfies the $\partial\bar\partial$-lemma. This in turn is equivalent by [DGMS75, Lemma 5.15] to all the maps in the diagram being isomorphisms. It ensues from this discussion that the $\partial\bar\partial$-property is invariant under pluripotential quasi-isomorphisms.

Proposition 3.8. *Let $f : A \to B$ be a pluripotential quasi-isomorphism. If one of the two bicomplexes is $\partial\bar\partial$, so is the other one.*

Proof. Without loss of generality, suppose that A is $\partial\bar\partial$. Consider the commutative diagram

$$\begin{tikzcd} H_{BC}(A) \arrow[r, "H_{BC}(f)"] \arrow[d, "i_A"'] & H_{BC}(B) \arrow[d, "i_B"] \\ H_{dR}(A) \arrow[r, "H_{dR}(f)"'] & H_{dR}(B) \end{tikzcd}$$

where *a priori* all the maps, besides i_B, are isomorphisms. It follows at once that i_B is an isomorphism as well, therefore B is $\partial\bar\partial$. □

Corollary 3.9. *A strongly formal CBBA is necessarily $\partial\bar\partial$.*

Corollary 3.10. *Let $f : A \to B$ is a bicomplex morphism. If A and B are $\partial\bar\partial$, then f is a pluripotential quasi-isomorphism if and only if $H_{dR}(f)$ is an isomorphism.*

Proof. In the commutative diagram

$$\begin{tikzcd} H_{BC}(A) \arrow[r, "H_{BC}(f)"] \arrow[d] & H_{BC}(B) \arrow[d] \\ H_{dR}(A) \arrow[r, "H_{dR}(f)"] \arrow[d] & H_{dR}(B) \arrow[d] \\ H_{A}(A) \arrow[r, "H_{A}(f)"] & H_{A}(B) \end{tikzcd}$$

the vertical arrows are isomorphisms. It follows immediately that $H_{BC}(f), H_A(f)$ are isomorphisms if and only if $H_{dR}(f)$ is one. □

3.2. **The Cartan model of a torus action.** Let X be a compact complex manifold with an action of $T = (\mathbb{S}^1)^n$ by biholomorphisms. Let $ET \to BT$ be a classifying bundle for T, e.g. $(\mathbb{S}^\infty)^n \to (\mathbb{CP}^\infty)^n$. Consider Borel's construction

$$\pi : X_T = (X \times ET)/T \to BT.$$

The Cartan model of this fibration is given by the CDGA

$$C_T(X, \mathbb{C}) = S_{\mathbb{C}}(\mathfrak{t}^\vee) \otimes A(X)^T$$

where $A(X)$ denote the smooth complex forms on X. It inherits a bigrading from the complex structure on X and is therefore a CBBA. Moreover, since T acts by biholomorphisms on X, it follows that T preserves the bigrading and $A(X)^T$ is in fact a CBBA. Lastly, recall that $\mathfrak{t}^\vee \cong \mathbb{R}^n$ as vector spaces with basis $\xi^1, \dots, \xi^n$, all of type $(1,1)$, and

$$S_{\mathbb{C}}(\mathfrak{t}^\vee) \cong \mathbb{C}[\xi^1, \dots, \xi^n] =: R.$$

The total differential is the map

$$d_T(1 \otimes \omega) = 1 \otimes d\omega - \sum_{i=1}^{n} \xi^i \otimes \iota_{\xi_i}\omega$$

which is the sum of the CBBA differentials

$$\partial_T(1 \otimes \omega) = 1 \otimes \partial\omega - \sum_{i=1}^{n} \xi^i \otimes \iota_{(\xi_i)^{0,1}}\omega,$$
$$\bar\partial_T(1 \otimes \omega) = 1 \otimes \bar\partial\omega - \sum_{i=1}^{n} \xi^i \otimes \iota_{(\xi_i)^{1,0}}\omega.$$

Remark 3.11. It is clear that the equivariant cohomology $H_T(X)$ inherits a bigrading from $C_T(X)$, since $H(C_T^{\bullet,\bullet}(X)) = H_T^{\bullet,\bullet}(X)$.

Lemma 3.12. *Let X be a compact complex $\partial\bar\partial$-manifold and G be a compact connected Lie group acting on X by biholomorphisms. Then the canonical injection of the CBBA $i : A(X)^G \to A(X)$ is a pluripotential quasi-isomorphism.*

Proof. There is an averaging map $a : A(X) \to A(X)^G$ given by $\omega \mapsto \int_G g^*\omega dg$ with respect to the normalized Haar measure. Again, since G acts by biholomorphisms, this map respects the bigradings and commutes with the differentials, therefore it is a bicomplex morphism. By construction, $a \circ i = \mathrm{id}$, thus the bicomplex $A(X)^G$ is a direct summand in $A(X)$.

Let us prove that $A(X)^G$ is $\partial\bar\partial$. If $\alpha \in A(X)^G$ is ∂-, $\bar\partial$- closed and d-exact, there is $\beta \in A(X)$ such that $\alpha = \partial\bar\partial\beta$, since $A(X)$ is $\partial\bar\partial$. But then $a(\beta) \in A(X)^G$ is a $\partial\bar\partial$-primitive of α. It follows that $A(X)^G$ is $\partial\bar\partial$.

Now, it is a classical result that a induces an isomorphism in de Rham cohomology (see, for example, [CE48, Theorem 12.1]). Since $H_{dR}(a)$ is an isomorphism and $A(X), A(X)^G$ are both $\partial\bar\partial$, we conclude by Corollary 3.10. □

3.3. **The case of complete smooth torics.** We present here the proof of [PSZ25] of strong formality for complete smooth toric varieties.

We recall briefly several properties of the equivariant cohomology of toric varieties. In the sequel the toric variety X is supposed complete smooth, unless mentioned otherwise. All the coefficients are complex.

By construction, $H_T(X)$ is an algebra over $H(BT) = \mathbb{C}[\xi^1, \dots, \xi^n] = R$, which we will rather note $R = \mathbb{C}[t_1, \dots, t_n]$ for convenience. Now, if $D_1, \dots, D_m$ are the torus-invariant divisors on X, denote by $\tau_i \in H_T^2(X)$ the image of 1 under the pushforward $H_T^0(D_i) \to H_T^2(X)$. The following proposition is the Stanley-Reisner presentation of a smooth toric variety's equivariant cohomology. For a complete treatment, the reader may consult [CLS11, Section 12.4].

Proposition 3.13. *Let $\mathcal{R} = \left\{ I \subset \{1, \dots, m\} \mid \bigcap_{i\in I} D_i = \varnothing \right\}$. Then*

$$H_T^{\bullet,\bullet}(X) = \mathbb{C}[\tau_1, \dots, \tau_m] \Big/ \left\langle \prod_{i \in I} \tau_i \mid I \in \mathcal{R} \right\rangle.$$

Now, recall that there is an R-module structure on $H_T(X)$, given by the pullback of the Borel fibration $\pi^* : R \to H_T(X)$

Lemma 3.14. *Let X be a complete smooth toric variety. Then there is a graded $\mathbb{C}$-algebras isomorphism*

$$H^\bullet(X) \cong H_T^\bullet(X)/\big(R^+ \cdot H_T^\bullet(X)\big)$$

Proof. By Corollary 2.21, X is equivariantly formal. In particular,

$$H_T(X) \cong H(X) \otimes R$$

as R-modules. But on the right handside, the image of the projection morphism, defined by $1 \otimes 1 \mapsto 1$ and $1 \otimes t_i \mapsto 0$ for all i, is simply $H(X)$. This corresponds on the left handside to $H_T(X)/(R^+ \cdot H_T(X))$. The projection morphism is a graded $\mathbb{C}$-algebra morphism inducing an isomorphism on the associated $\mathbb{C}$-vector spaces, therefore it is a $\mathbb{C}$-algebra isomorphism. ☐

Remark 3.15. Notice that $H_T(X)/(R^+ \cdot H_T(X))$ carries a natural bigrading, namely

$$\big(H_T(X)/(R^+ \cdot H_T(X))\big)^{p,q} = H_T^{p,q}(X)/(R^+ \cdot H_T(X))^{p,q}.$$

Moreover, if a bicomplex's associated total complex has null differential $\partial + \bar\partial = 0$, then it has itself null differentials $\partial = \bar\partial = 0$. This follows from the fact that a pure type element $a^{p,q}$ which is d-closed is necessarily ∂- and $\bar\partial$-closed. It follows at once that the bicomplex $(H_T(X)/R^+ \cdot H_T(X))^{\bullet,\bullet}$ has trivial differentials.

Example 3.16. Let us compute the equivariant cohomology of $\mathbb{CP}^1$. It is a toric variety of rank 1, with the torus $T = \mathbb{S}^1$ acting by

$$t \cdot [z_0 : z_1] = [z_0 : tz_1], \qquad \text{for } t \in \mathbb{S}^1 \text{ and } [z_0 : z_1] \in \mathbb{CP}^1.$$

Its classifying space is $\mathbb{CP}^\infty$, thus $R = H^{\bullet,\bullet}(BT) = \mathbb{C}[u]$ with u of type $(1,1)$.

There are just two torus invariant divisors, namely $[1:0]$ and $[0:1]$. This will induce only one non-trivial relation in its Stanley-Reisner ideal. Concretely, if τ_0 corresponds to $[1:0]$ and τ_1 corresponds to $[0:1]$, then by Proposition 3.13

$$H_T(X) \cong \mathbb{C}[\tau_0, \tau_1]/(\tau_0\tau_1).$$

Lastly, the R-module structure on $H_T(X)$ comes from the *equivariant divisor relation*, namely

$$u \mapsto \tau_1 - \tau_0.$$

We refer the reader to [CLS11, Section 12.4] for more details.

Notice however, that given this R-module structure, $H_T(X)$ is a free R-module of finite type

$$H_T(X) = R \cdot \langle 1, \tau_0 \rangle$$

for example. One can then write $\tau_1 = u + \tau_0$. Then

$$H_T(X) \cong \mathbb{C}[u, \tau_0]/(\tau_0(1 + \tau_0)).$$

Remark, by the same token, that $H_T(X)/(R^+ H_T(X)) = \mathbb{C}[u, \tau_0]/(\tau_0(1+\tau_0), u) \cong \mathbb{C}[\tau_0]/(\tau_0(1+\tau_0)) \cong H(X)$.

Lemma 3.17. [PSZ25, Lemma 3.2.2] *There exist real forms $\alpha_i \in C_T(X)^{1,1}$ such that $\tau_i = [\alpha_i]$. One may choose them supported arbitrarily closely along D_i, in particular one may assume that the CBBA product $\prod_{i \in I} \alpha_i = 0$ for $I \in \mathcal{R}$.*

Remark 3.18. In particular, Lemma 3.17 implies that there is a pluripotential quasi-isomorphism $f : H_T(X) \to C_T(X)$ defined by $f(\tau_i) = \alpha_i$ for all i.

Fix such a pluripotential quasi-isomorphsim $f : H_T(X) \to C_T(X)$. Set $g = f \circ \pi^* : R \to C_T(X)$ and notice that g induces an R-module structure on $C_T(X)$ for which f is trivially R-linear.

Lastly, define a CBBA

$$S = \Lambda(s_i, \partial s_i, \bar{\partial} s_i, \partial\bar{\partial} s_i, i = 1, \dots, n)$$

with s_i of type $(0,0)$ and an R-module structure induced by $t_i \mapsto i\partial\bar{\partial} s_i$ for all i.

Lemma 3.19. *With the notations above, there is a zig-zag of pluripotential quasi-isomorphisms between $C_T(X) \otimes_R S$ and $H_T(X)/(R^+ \cdot H_T(X))$, a bicomplex with trivial differentials.*

Proof. Firstly, we claim that the map

$$f \otimes \mathrm{id} : H_T(X) \otimes_R S \to C_T(X) \otimes_R S$$

is a pluripotential quasi-isomorphism. This can be seen, for example, as a consequnce of the fact that $C_T(X)$ is $\partial\bar{\partial}$ [PSZ25, Lemma 3.1.5]. The bicomplexes $H_T(X)$ and S, on the other hand, are trivially $\partial\bar{\partial}$. On the other hand, $H_{dR}(f \otimes \mathrm{id})$ is an isomorphism by Künneth's formula for de Rham cohomology. But then, by Corollary 3.10 this is a pluripotential quasi-isomorphism.

On the other hand, we claim that the bicomplex morphism

$$\psi : H_T(X) \otimes_R S \to H_T(X)/(R^+ \cdot H_T(X))$$

defined by $1 \otimes 1 \mapsto 1$ and $1 \otimes s_i \mapsto 0$ is a pluripotential quasi-isomorphism. Let us prove first that $H_{BC}(\psi)$ is an isomorphism. Firstly, recall that by Theorem 2.18, $H_T(X)$ is free over R, hence may be written as

$$H_T(X) = \bigoplus_{j \in J} R \cdot e_j$$

for a suitable index set J and elements $e_j \in H_T(X)$. But then

$$H_T(X) \otimes_R S = \bigoplus_{j \in J} (R \otimes_R S) \cdot e_j \cong \bigoplus_{j \in J} S \cdot e_j.$$

This is a bicomplex direct sum, since the differentials act non-trivially only on S

$$\partial(S \cdot e_j) \subset S \cdot e_j, \qquad \bar{\partial}(S \cdot e_j) \subset S \cdot e_j.$$

But H_{BC} commutes with direct sums, therefore

$$H_{BC}(H_T(X) \otimes_R S) = H_{BC}\Big(\bigoplus_{j \in J} S e_j\Big) = \bigoplus_{j \in J} H_{BC}(S) \cdot [e_j] = \bigoplus_{j \in J} k \cdot [e_j]$$

since S is contractible.

On the other hand, the Bott-Chern cohomology of $(H_T(X)/R^+ \cdot H_T(X))$ is the bicomplex itself, since the differentials are null. But then, since $M/IM \cong M \otimes_R R/I$, it follows

$$H_T(X)/(R^+ \cdot H_T(X)) \cong H_T(X) \otimes_R (R/R^+) \cong \bigoplus_{j \in J} (R/R^+) \cdot \bar{e}_j \cong \bigoplus_{j \in J} k \cdot \bar{e}_j,$$

where $\bar{e}_j$ is the equivalence class of $e_j \in H_T(X)$ in $H_T(X)/(R^+ \cdot H_T(X))$. Now ψ sends $e_j \otimes 1$ to $e_j \psi(1 \otimes 1) = \bar{e}_j$, therefore $H_{BC}(\psi)$ sends a basis to another and preserves the bigradings. It ensues that $H_{BC}(f)$ is an isomorphism.

By Lemma 3.14, there is a zig-zag of Bott-Chern weak equivalences, that is bicomplex morphisms that induce isomorphisms in Bott-Chern

$$C_T(X) \otimes_R S \xleftarrow{f \otimes \mathrm{id}} H_T(X) \otimes_R S \xrightarrow{\psi} H_T(X)/(R^+ \cdot H_T(X)).$$

All of these bicomplexes being $\partial\bar{\partial}$, it follows that the maps above are pluripotential quasi-isomorphisms. ☐

Lemma 3.20. *With the notations above, there is a zig-zag of pluripotential quasi-isomorphisms between $C_T(X) \otimes_R S$ and $A(X)$.*

Proof. Recall that $C_T(X)$ has been equipped with an R-module structure, induced by $g = f \circ \pi^* : R \to C_T(X)$. This amounts to saying that in $C_T(X) \otimes_R S$, one has $g(t_i) \otimes 1 = 1 \otimes i\partial\bar{\partial}s_i$, which by abuse of notation shall be written $g(t_i) = i\partial\bar{\partial}s_i$. There is also the canonical injection morphism from R to $C_T(X) = R \otimes A(X)^T$ that sends r to $r \otimes 1$. The morphism g, by construction, induces the same map in Bott-Chern cohomology as the canonical injection. It follows that $g(t_i) - t_i = i\partial\bar{\partial}\eta_i$ for suitable $\eta_i \in A(X)^{0,0}$. Define then an automorphism

$$\begin{aligned} \varphi : C_T(X) \otimes_R S &\to C_T(X) \otimes_R S \\ 1 \otimes s_i &\mapsto 1 \otimes (s_i - \eta_i) \end{aligned}$$

then $\partial' = \varphi^{-1}\partial\varphi$ and $\bar{\partial}' = \varphi^{-1}\bar{\partial}\varphi$. One has

$$\begin{aligned} i\partial'\bar{\partial}'s_i = i\varphi^{-1}\partial\bar{\partial}\varphi(s_i) = i\varphi^{-1}\partial\bar{\partial}(s_i - \eta_i) = \\ = \varphi^{-1}\Big(g(t_i) - \big(g(t_i) - t_i\big)\Big) = \varphi^{-1}(t_i) = t_i \end{aligned}$$

as φ is the identity map on $C_T(X)$. Thus $(C_T(X) \otimes_R S, \partial, \bar{\partial})$ and $(C_T(X) \otimes_R S, \partial', \bar{\partial}')$ are isomorphic bicomplexes, but in the latter one, one has $t_i = i\partial'\bar{\partial}'s_i$, whence it may be seen as an extension

$$C_T(X) \otimes_R S = \big(R \otimes A(X)^T\big) \otimes_R S = (R \otimes_R S) \otimes A(X)^T \cong S \otimes A(X)^T$$

of $A(X)^T$ by the contractible CBBA S. All the involved bicomplexes are $\partial\bar{\partial}$, therefore the Künneth formula holds in Bott-Chern cohomology. The projection $p : S \otimes A(X)^T \to A(X)^T$ is then a Bott-Chern weak equivalence, thus a pluripotential quasi-isomorphism since both the complexes are $\partial\bar{\partial}$, whereas $i : A(X)^T \hookrightarrow A(X)$ is a pluripotential quasi-isomorphism by Lemma 3.12.

Consequently there is a zig-zag of pluripotential quasi-isomorphisms

$$C_T(X) \otimes S \xrightarrow{\cong} A(X)^T \otimes S \;=\!=\!=\; A(X)^T \otimes S \xrightarrow{i \circ p} A(X).$$

☐

Theorem 3.21. *A complete smooth toric variety X is strongly formal.*

Proof. This follows as an immediate corollary of Lemma 3.19 and Lemma 3.20. Indeed, one has a pluripotential zig-zag

$$\begin{tikzcd} C_T(X) \otimes_R S & & & H_T(X) \otimes_R S \arrow[lll, "f \otimes \mathrm{id}"'] \arrow[r, "\psi"] & H_T(X)/(R^+ \cdot H_T(X)) \\ C_T(X) \otimes_R S \arrow[u, "\mathrm{id}"] \arrow[r, "\cong"] & A(X)^T \otimes S & A(X)^T \otimes S \arrow[l, "\mathrm{id}"'] \arrow[rr, "i \circ p"] & & A(X). \end{tikzcd}$$

connecting $A(X)$ to a CBBA with trivial differentials. ☐

3.4. **The case of compact homogeneous Kähler manifolds.** Following [PSZ25], we show strong formality of compact homogeneous Kähler manifolds.

3.4.1. *Complete intersection type algebras.*

Definition 3.22. A graded algebra $H^\bullet$ is said to be of complete intersection type if we can write

$$H = k[x_1, \dots, x_n]/(r_1, \dots, r_m),$$

with x_i of positive even degree and with $r_1, \dots, r_m$ a *regular sequence* in $k[x_1, \dots, x_n]$, meaning that r_{i+1} is a nonzerodivisor in $k[x_1, \dots, x_n]/(r_1, \dots, r_i)$ for any i in $\{0, \dots, m-1\}$.

For a bigraded algebra, one has a similar definition.

Definition 3.23. A bigraded algebra $H^{\bullet,\bullet}$ is said to be of complete intersection type if (3.22) holds with $|x_i| = (p_i, p_i)$, for positive p_i, and with $r_1, \dots, r_m \in k[x_1, \dots, x_n]$ a regular sequence.

There is a known formality result for CDGAs.

Lemma 3.24. *Let $H^\bullet = k[x_1, \dots, x_n]/(r_1, \dots, r_m)$ be of complete intersection type.*

(i) Let V be a a graded space with basis $p_1, \dots, p_m$ and $|p_i| = |r_i| - 1$. Consider the CDGA

$$K = k[x_1, \dots, x_n] \otimes \Lambda(V)$$

with differential d defined by $dx_i = 0$, $dp_i = r_i$. Then the CDGA morphism $K \to H$, sending x_i to x_i, p_i to 0, is a quasi-isomorphism, provided H is endowed with the trivial differential.

(ii) Let A be a CDGA with cohomology H. Then A is formal.

There is the following CBBA analogue.

Lemma 3.25. *Let $H^{\bullet,\bullet} = \mathbb{C}[X_1, \dots, X_n]/(R_1, \dots, R_m)$ be of complete intersection type.*

(i) Let V be a bigraded vector space with basis $(P_j,\ \partial P_j,\ \bar\partial P_j,\ j = 1, \dots, m)$ such that $|P_j| = |R_j| - (1,1)$. Consider the CBBA $K = \mathbb{C}[X_1, \dots, X_n] \otimes \Lambda(V)$ with

$$\begin{aligned} dX_i &= 0, \\ dP_j &= \partial P_j + \bar\partial P_j, \\ i\partial\bar\partial P_j &= R_j. \end{aligned}$$

Consider $H^{\bullet,\bullet}$ as a CBBA with trivial differentials. Then the CBBA morphism $\varphi : K \to H$ which sends X_i to X_i, P_j to 0, is a pluripotential quasi-isomorphism.

(ii) Let $A^{\bullet,\bullet}$ be a CBBA satisfying the $\partial\bar\partial$-lemma. If A has cohomology H, then A is strongly formal.

Proof. (i) Notice that H and K are first-quadrant. As such, if $\varphi : H \to K$ induces isomorphisms $H_\partial(\varphi)$ and $H_{\bar\partial}(\varphi)$, then φ is a pluripotential quasi-isomorphism. We shall prove that $H_{\bar\partial}(\varphi)$ is an isomorphism, the other argument being obtained by interchanging ∂ and $\bar\partial$.

Now, consider the forget functor

$$U_{\bar\partial} : \mathbf{CBBA} \to \mathbf{CDGA}$$

that, at the object level, sends $(A^{\bullet,\bullet},\, \partial,\, \bar\partial)$ to $(A^\bullet,\, \bar\partial)$. Notice that in $U_{\bar\partial}K$, the elements ∂P_j are non-exact. One can think of it as two successive extensions of $\mathbb{C}[X_1,\dots,X_n]$, namely

$$U_{\bar\partial}K = \big(\mathbb{C}[X_1,\dots,X_n] \otimes_{\bar\partial} \Lambda(P_j, \bar\partial P_j, j=1,\dots,m)\big) \otimes_{\bar\partial} \Lambda(\partial P_j, j=1,\dots,m)$$

the first extension being defined with the obvious differential, the second satisfying $\bar\partial(i\partial P_j) = -R_j$. The restriction of $U_{\bar\partial}(\varphi)$ to $\Lambda(\partial P_j, j = 1,\dots,m)$ is zero, while the CBBA itself is contractible. It thus suffices to show that $U_{\bar\partial}(\varphi)$ restricted to $\mathbb{C}[X_1,\dots,X_n] \otimes_{\bar\partial} \Lambda(P_j, \bar\partial P_j, j=1,\dots,m)$ is a $H_{\bar\partial}$-quasi-isomorphism. But this follows immediately from Lemma 3.24 (i).

(ii) We shall construct a pluripotential quasi-isomorphism $\Phi : K \to A$. In order to do this, it suffices to choose Bott-Chern representatives as images for the generators X_i and P_j. Recall that the X_i have type (p_i, p_i), so one needs to choose for $\Phi(X_i)$ representatives of the same type. That these should exist is a consequence of the hypothesis that A is $\partial\bar\partial$, hence a cohomology class of pure type always has a representative of the same pure type. Note that a de Rham representative of type (p_i, p_i) is automatically a Bott-Chern representative of the same type.

Obviously $\Phi(R_j)$ is a Bott-Chern representative, because it is a polynomial in Bott-Chern representatives. Moreover, it is $i\partial\bar\partial$-exact, as it has by definition the same Bott-Chern cohomology class as R_j, which by (i) is cohomologous to zero. Let Q_j be an element such that $\Phi(R_j) = i\partial\bar\partial Q_j$. Then define $\Phi(P_j) = Q_j$ and extend Φ to all of K by linearity and Leibniz's rule. By construction, Φ is a Bott-Chern weak equivalence.

Lastly, by a Koszul-type argument, it follows that K is $\partial\bar\partial$. Since Φ is a Bott-Chern equivalence between two $\partial\bar\partial$ bicomplexes, it is a pluripotential quasi-isomorphism. □

3.4.2. *Compact homogeneous Kähler manifolds.* We deduce the main theorem of this subsection as a corollary of the last lemma.

Theorem 3.26. *A compact homogeneous Kähler manifold X is strongly formal.*

Proof. Write $X = T \times F$, where T is a torus and F is a generalized flag manifold, both compact Kähler, both $\partial\bar\partial$ *a fortiori*, thus Künneth holds in Bott-Chern cohomology. It then suffices to show that each of the factors is strongly formal.

A torus is strongly formal, because its cohomology is free, trivially of complete intersection type. On the other hand, it is a well-known result that the de Rham cohomology of a generalized flag manifold is of complete intersection type. □

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