1) Derivations, differentials

R ring $\qquad A$ an R-algebra $\qquad M \in (A\text{-Mod})$

R-derivation : $d : A \longrightarrow M$ st $\begin{cases} \forall f,g \in A, \quad d(fg) = f dg + g df & ① \\ d \text{ is } R\text{-linear} \\ d(a) = 0 \quad \text{for } a \in R & ② \end{cases}$

$\mathrm{Der}_R(A,M)$ is an A-module

Ex: • $A = R[X_1,\dots,X_n]$ $\qquad \mathrm{Der}_R(A,A) = A\frac{\partial}{\partial X_1} \oplus \cdots \oplus A\frac{\partial}{\partial X_n}$

• $R = k$ $\qquad A = \frac{k[X_1,\dots,X_n]}{I}$ $\qquad a : \mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, A =: X$

$\hookrightarrow$ k is an A-module via evaluation at a

$$\mathrm{Der}_k(A,k) = T_a(X) = \left(\mathfrak{m}_a / \mathfrak{m}_a^2\right)^*$$

Defn: $\Omega^1_{A/R}$ is an A-module with a derivation $d : A \longrightarrow \Omega^1_{A/R}$

st:

$$\begin{tikzcd} A \arrow[r, "\forall \partial"] \arrow[dr, "d"'] & M \\ & \Omega^1_{A/R} \arrow[u, dotted, "\exists !"'] \end{tikzcd}$$

$$\mathrm{Hom}_A(\Omega^1_{A/R}, M) \simeq \mathrm{Der}_R(A,M)$$

Construction: $\Omega^1_{A/R} = \dfrac{\text{free } A\text{-module on } da, a \in A}{\text{relations } ① \text{ and } ②}$

Rem: $\mathrm{Der}_R(A,A) = (\Omega^1_{A/R})^\vee = \mathrm{Hom}_A(\Omega^1_{A/R}, A)$

Ex: $A = R[X_1,\dots,X_n]$ $\qquad \Omega^1_{A/R} \simeq A dX_1 \oplus \cdots \oplus A dX_n$

Prop (exact sequences): Let $\varphi: A \to B$ morphism of R-algebras

(1) $$\Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} \longrightarrow 0$$

(2) If $B = A/I$, then $\Omega^1_{B/A} = 0$, and there is an exact sequence

$$\begin{array}{ccccccc} I/I^2 & \longrightarrow & \Omega^1_{A/R} \otimes_A B & \longrightarrow & \Omega^1_{B/R} & \longrightarrow & 0 \\ f & \longmapsto & df \otimes 1 & & & & \end{array}$$

Proof: easy: prove that the $\mathrm{Hom}_B(-, M)$ sequences are exact $\forall M \in (B\text{-Mod})$.

Ex: $B = \dfrac{R[X_1, \dots, X_n]}{I}$, $A = R[X_1, \dots, X_n]$

By (2), $$\Omega^1_{B/R} = \frac{B\,dX_1 \oplus \cdots \oplus B\,dX_n}{\langle df,\ f \in I \rangle}$$

also, $\forall f \in B$, $df = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i$ in $\Omega^1_{B/R}$

$$\Omega^1_{B/R} \simeq \mathrm{Coker}\, J^T \quad \text{where} \quad J = \left(\frac{\partial f_i}{\partial X_j}\right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$$

Prop: $R \to A$. Let $I = \mathrm{Ker}\begin{pmatrix} A \otimes_R A \longrightarrow A \\ f \otimes g \longmapsto fg \end{pmatrix} \subset A \otimes_R A$

then: (1) I/I^2 is an A-module, and

(2) $I/I^2 \simeq \Omega^1_{A/R}$ (check that $\begin{array}{l} \delta: A \to I/I^2 \\ f \mapsto 1 \otimes f - f \otimes 1 \end{array}$ is a universal derivation)

Globalization

Let $X \to S$ be a morphism of schemes

- If X is separated over S, then $\Delta: X \to X \times_S X$ is a closed immersion. Let $\mathcal{J}$ be its sheaf of ideals.

$\Omega^1_{X/S} := \mathcal{J}/\mathcal{J}^2$ viewed as an $\mathcal{O}_X$-module.

- In general, let W be the largest open subset of $X \times X$ in which $\Delta(X)$ is closed, $\mathcal{J} :=$ sheaf of ideals of $X \xrightarrow[\Delta]{} W$, and $\Omega^1_{X/S} := \mathcal{J}/\mathcal{J}^2$.

$\Omega^1_{X/S}$ is a quasicoherent sheaf over X.

If X and S are affine, $\Omega^1_{X/S} = \Omega^1_{\Gamma(X)/\Gamma(S)}$.

Prop (exact sequences): If $X \xrightarrow{\varphi} Y \longrightarrow S$ morphism of schemes,

(1) $\varphi^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$

(2) if φ is a closed immersion defined by $\mathcal{J}$, then

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \varphi^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0 \quad \text{and} \quad \Omega^1_{X/Y} = 0$$

Prop: (base change)

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ S' \arrow[r, "\varphi"'] & S \end{tikzcd}$$

then $\Omega^1_{X'/S'} \simeq \varphi'^*\Omega^1_{X/S}$

$$\begin{tikzcd} X_1 \times_S X_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X_1 \arrow[d] \\ X_2 \arrow[r] & S \end{tikzcd}$$

then $\Omega^1_{X_1 \times_S X_2 / S} \simeq \left(p_1^*\Omega^1_{X_1/S}\right) \oplus \left(p_2^*\Omega^1_{X_2/S}\right)$

Nonsingular varieties. Assume k is a perfect field.

Let X be a finite type affine scheme $/k$. $X = \operatorname{Spec} \dfrac{k[X_1, \dots, X_n]}{(f_1, \dots, f_r)} = \operatorname{Spec} A$

$J_X := \left(\dfrac{\partial f_i}{\partial X_j}\right)_{\substack{i=1,\dots,r \\ j=1,\dots,n}}$ jacobian

Let $a : \operatorname{Spec} k \longrightarrow X$ a point.

$$T_a(X) \simeq \left(\mathfrak{m}_a/\mathfrak{m}_a^2\right)^* \simeq \operatorname{Ker}(J_X(a))$$

$\mathfrak{m}_a$ maximal ideal of $\mathcal{O}_{X,a}$

Definition: X is non singular at a if $\dim T_a(X) = \dim_a(X) = \dim \mathcal{O}_{X,a}$

Theorem: TFAE: (1) X is non singular at a

(2) $\operatorname{rk} J_X(a) = n - \dim_a(X)$ (Jacobi criterion)

(3) $\mathcal{O}_{X,a}$ is a regular local ring

(4) $\left(\Omega^1_{X/k}\right)_a$ is free over $\mathcal{O}_{X,a}$, of rank $\dim_a(X)$.

(5) $\exists\, U$ neighborhood of a st $\Omega^1_{X/k}\big|_U$ is free of rank $\dim_a(X)$

(6) $\dim\left(\Omega^1_{X/k} \otimes_A k(a)\right) \leq \dim_a(X)$ (and it is actually equal)

$k(a)$: residue field

(7) $\forall$ diagram

$$\begin{tikzcd} & & X \arrow[d] \\ T \arrow[r] \arrow[urr, bend left] & T_1 \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k \end{tikzcd}$$

where
- $T_1 = \operatorname{Spec} B$, B Artin local k-alg
- $T \hookrightarrow T_1$ closed immersion by $I \subset B$ with $I^2 = 0$
- the closed pt of T maps to a

there exists dotted arrow.

Idea of proof: (1) $\Leftrightarrow$ (2) rank theorem

(1) $\Leftrightarrow$ (3) if A is a local Noetherian ring, $\dim A \leq \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ with equality if the ring is regular (by defn)

(3) $\Leftrightarrow$ (6)

$$\Omega^1_{X/k} \simeq \operatorname{Coker}\left(A^r \xrightarrow{J_X^T} A^n\right)$$

$$\Rightarrow \Omega^1_{X/k} \otimes_A k(a) \simeq \operatorname{Coker} J_X(a)^T$$

(4) $\Leftrightarrow$ (5) localization

(7) $\Rightarrow$ (4) will be proved more generally next.

(7) $\Rightarrow$ (1) case $A = \dfrac{k[X,Y]}{(f)}$

(1) $\Leftrightarrow$ $\dfrac{\partial f}{\partial X}(a) \neq 0$ or $\dfrac{\partial f}{\partial Y}(a) \neq 0$

WLOG assume $a = (0,0)$

If X is singular at a, then $f \in (X,Y)^2$

$$f = f_d + f_{d+1} + \cdots + f_p$$

f_i homogeneous, $d \geq 2$, $f_d \neq 0$

Take $B = \dfrac{k[X,Y]}{(X,Y)^{d+1}}$, $I = (X,Y)^d$

$$\begin{tikzcd}
 & \overset{X}{\dfrac{k[X,Y]}{(f)}} \arrow[dl, "X \mapsto X,\ Y \mapsto Y"'] \arrow[d, dashed, "\nexists \text{ because } f_d \neq 0"] \\
\underset{T}{\dfrac{k[X,Y]}{(X,Y)^d}} & \underset{T_1}{\dfrac{k[X,Y]}{(X,Y)^{d+1}}} \arrow[l, two heads]
\end{tikzcd}$$