Nikola - Betti - log de Rham comparison

X = analytic log variety that is fs and such that X admits an "affine" atlas $\bigcup U_\lambda = X$ where $U_\lambda = \mathrm{Spec}\left(P_\lambda \to \frac{\mathbb{C}[P_\lambda]}{(\Sigma_\lambda)}\right)$ with P_λ fs, Σ_λ ideal of P_λ.

<u>Theorem</u>: $H^q(X, \omega_X^\cdot) \simeq H^q(X^{\log}; \mathbb{C})$

$\nwarrow$ Kato-Nakayama space

Recall: $\tau : X^{\log} \longrightarrow X$; $\omega_X^\cdot$ is the de Rham complex with log forms.

<u>Lemma</u>: $R^q \tau_* \underline{\mathbb{C}} \simeq \Lambda^q(\mathcal{M}_X^{gp}/\mathcal{O}_X^\times) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$

<u>Proof</u>: $\mathcal{M}_X^{gp}/\mathcal{O}_X^\times \longrightarrow R^1\tau_* \mathbb{Z}$?

Start with the exponential sequence $0 \to \mathbb{Z} \to \tau^{-1}\mathcal{O}_X \to \tau^{-1}\mathcal{O}_X^\times \to 0$

[we'll go back to this later.] Key: τ is proper, use base change.

<u>The sheaf $\mathcal{L}$</u>: (on $\mathcal{O}_X^{\log}$)

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z} & \xrightarrow{(2\pi)} & \mathcal{C}(-, i\mathbb{R}) & \xrightarrow{\exp} & \mathcal{C}(-, S^1) & \to & 0 \\
 & & \| & & \uparrow & & \uparrow c & & \\
0 & \to & \mathbb{Z} & \to & \mathcal{L} & \xrightarrow[\exp]{\urcorner} & \tau^{-1}\mathcal{M}_X^{gp} & \to & 0
\end{array}$$

$\mathcal{C}$ = continous functions

$c(m) : (x, h) \mapsto h(m_x)$

[Recall: $X^{\log} = \{(x,h),\ x \in X,\ h : \mathcal{M}_{X,x}^{gp} \to S^1 \mid \forall f \in \mathcal{O}_{X,x}^\times,\ h(f) = \frac{f(x)}{|f(x)|}\}$]

<u>The sheaf $\mathcal{O}_X^{\log}$</u>:

We have $\tau^{-1}\mathcal{O}_X \xrightarrow{h} \mathcal{L}$, take $\mathcal{O}_X^{\log} := \tau^{-1}\mathcal{O}_X \otimes \mathrm{Sym}(\mathcal{L}) / (f \otimes 1 = 1 \otimes h(f))$

<u>Lemma:</u> If $y \in X^{\log}$, $y = (x, -)$; $(t_i) \in \mathcal{L}_y$ s.t. $(\exp(t_i))$ is a basis of $\mathcal{M}^{gp}_{X,x}/\mathcal{O}^\times_{X,x}$.
Then
$$\mathcal{O}^{\log}_{X,y} \simeq \mathcal{O}_{X,x}[t_1, \dots, t_n]$$

(<u>Proof</u>: $0 \to \tau^{-1}\mathcal{O}_X \to \mathcal{L} \to \tau^{-1}(\mathcal{M}^{gp}_X/\mathcal{O}^\times_X) \to 0$) $\lrcorner$

Filtration on $\mathcal{O}_X^{\log}$: $\mathrm{Fil}^n \mathcal{O}_X^{\log} = \tau^{-1}\mathcal{O}_X \otimes \mathrm{Sym}^{\leq n}(\mathcal{L}) / \dots$

<u>Lemma:</u> $gr^n(\mathcal{O}_X^{\log}) \simeq \tau^{-1}\mathcal{O}_X \otimes \mathrm{Sym}^n(\mathcal{M}^{gp}_X/\mathcal{O}^\times_X)$.

Induced by $\tau^{-1}(\mathcal{M}^{gp}_X/\mathcal{O}^\times_X) \simeq \mathcal{L}/\tau^{-1}\mathcal{O}_X \hookrightarrow gr^1$

Now define
$$\omega_X^{\cdot,\log} := \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega_X^\cdot)$$

<u>log Poincaré lemma:</u> $\mathbb{C}_{X^{\log}} \xrightarrow{\sim} \omega_X^{\cdot,\log}$

<u>Theorem:</u> $\omega_X^\cdot \xrightarrow{\sim} R\tau_* \omega_X^{\cdot,\log}$ $(\simeq R\tau_* \mathbb{C})$