Julien - Variétés logarithmiques / log schemes
Schémas

1) Monoids
2) Log schemes
3) Operations on log schemes
4) Charts

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1) Monoids

<u>Def</u>: Mon = the category of (commutative + unital) monoids $(Q, \cdot, 1)$

Mon admits ~~filtered colimits~~ projective limits and forget: Mon $\to$ Set is ~~co~~continuous

Mon admits ~~limits~~ colimits

- Product & coproduct are the same $Q_1 \oplus Q_2 := Q_1 \times Q_2$
- Non-trivial: coequalizers (quotients)

  $E \subset Q \times Q$ is a <u>congruence relation</u> if it is both a congruence relation and a monoid.

  $\rightsquigarrow$ Q/E is a monoid.

  For $f : Q \to Q'$ a surjective morphism of monoids, $E_f := \{ (x,y) \in Q \times Q \mid f(x) = f(y) \}$ is a congruence relation, and $Q/E_f \xrightarrow{\sim} Q'$.

  If E, E' are congruence relations then $E \cap E'$ is. So for $S \subset Q \times Q$ we can speak of the congruence relation E_S generated by S, and take the quotient.

  $\mathrm{Coeq}\left(Q \underset{u_2}{\overset{u_1}{\rightrightarrows}} Q' \right) = Q'/E_S$, S generated by $\{ (u_1(x), u_2(x)) , x \in Q \}$.

Pushout square:

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & Q_1 \arrow[d] \\
Q_2 \arrow[r] & Q_1 \oplus_Q Q_2 = (Q_1 \oplus Q_2)/E
\end{tikzcd}$$

<u>Lemma</u>: If Q_1 or Q_2 is a group then $(x_1, x_2) \underset{E}{\sim} (x_1', x_2')$ iff

$\exists\, a, b \in Q$ st $\begin{cases} x_1 + u_1(b) = x_1' + u_1(b) \\ x_2 + u_2(a) = x_2' + u_2(a) \end{cases}$

2) log schemes

<u>Def</u>: X a scheme. A <u>pre-log structure</u> on X is $(\mathcal{M}, \alpha)$ where $\mathcal{M}$ is a sheaf of monoids and $\alpha : \mathcal{M} \to \mathcal{O}_X$.

It is a <u>log structure</u> if $\alpha^{-1}(\mathcal{O}_X^\times) \longrightarrow \mathcal{O}_X^\times$ is an isomorphism.

A <u>morphism</u> of (pre-)log structures is the obvious thing:
$$\begin{tikzcd} \mathcal{M} \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & \mathcal{M}' \arrow[dl, "\alpha'"] \\ & \mathcal{O}_X & \end{tikzcd}.$$

$\rightsquigarrow$ Categories Log_X, Prelog_X.

<u>Lemma</u>: The inclusion functor $\mathrm{Log}_X \hookrightarrow \mathrm{Prelog}_X$ has a left adjoint $a : \mathrm{Prelog}_X \to \mathrm{Log}_X$.
$(\mathcal{M}, \alpha) \longmapsto (\mathcal{M}^a, \alpha^a)$

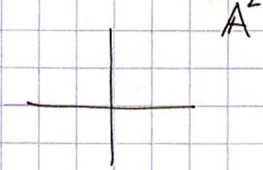
$$\begin{tikzcd} \alpha^{-1}(\mathcal{O}_X^\times) \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathcal{O}_X^\times \arrow[r] & \mathcal{M}^a \arrow[ul, phantom, "\ulcorner" near start] \end{tikzcd}$$
(pushout square)

α^a is given by the universal property using
$$\begin{tikzcd} & \mathcal{M} \arrow[d] \arrow[dd, bend left, "\alpha"] \\ \mathcal{O}_X^\times \arrow[r] \arrow[dr, hook, bend right] & \mathcal{M}^a \arrow[d, dashed, "\alpha^a"] \\ & \mathcal{O}_X \end{tikzcd}$$

<u>Example 1</u>: X scheme, $D \subset X$ a divisor
$\mathcal{M}_D : U \mapsto \{ f \in \mathcal{O}_X(U) \mid f|_{U \setminus D} \in \mathcal{O}_X^\times(U \setminus D) \}$

$\alpha : \mathcal{M}_D \hookrightarrow \mathcal{O}_X$.

<u>Sub-example</u>: $X = \mathbb{A}^n = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$, $D = V(x_1 \cdots x_n)$



$\mathcal{O}_X^\times(X) = \mathbb{C}^* \overset{\alpha}{\hookleftarrow} \mathcal{M}_D(X) = \{ f \in \mathbb{C}[x_1, \dots, x_n] \mid f \text{ is invertible in } \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \}$

$= \mathbb{C}^* x_1^{\mathbb{N}} \cdots x_n^{\mathbb{N}}$.

<u>Example 0</u>: the trivial log structure $\mathcal{M} = \mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$.

[Also: the <u>null</u> log structure $\mathcal{M} = \mathcal{O}_X$.]

<u>Def</u>: Q a monoid, Q^{gp} its group completion, $Q^\times \subset Q$ the invertibles.
Q is

* integral if $Q \to Q^{gp}$ is injective, i.e. $a+e = b+e \Rightarrow a = b$.
* fine if integral + finitely generated
* saturated if integral + $\left(\begin{array}{c}\forall a \in Q^{gp} \\ \forall n \in \mathbb{N}^*\end{array}, \ na \in Q \Rightarrow a \in Q \right)$
* toric if fine, saturated, and Q^{gp} is torsion-free ($\simeq \mathbb{Z}^n$).

<u>Ex</u>: σ a strictly convex rational polyhedral cone, $Q = S_\sigma$ is toric.

<u>Ex. 2</u>: Q a monoid, $A_Q = \mathrm{Spec}(\mathbb{C}[Q])$

$\mathcal{M}_Q := Q$, $Q \xrightarrow{\alpha} \mathbb{C}[Q]$ ($\mathcal{M}_Q$ is the constant sheaf of monoids)

$(A_Q, \mathcal{M}_Q, \alpha)$ is a pre-log structure $\rightsquigarrow (A_Q, \mathcal{M}_Q^a, \alpha^a)$.

If Q is toric, $A_{Q^{gp}} = \mathrm{Spec}(\mathbb{C}[Q^{gp}]) \simeq \mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$.

$A_{Q^{gp}} \overset{j}{\hookrightarrow} A_Q$, $D := A_Q \setminus A_{Q^{gp}}$, it is a divisor.

$\rightsquigarrow (\mathcal{M}_D, \alpha_D)$ log structure on A_Q.

They are the same log structure on A_Q! We check that the global functions agree.

- For $(\mathcal{M}_D, \alpha_D)$: $\mathbb{C}^* \oplus Q$.
- For $(A_Q, \mathcal{M}_Q^a, \alpha^a)$:

Pushout square:

$$\begin{tikzcd} Q^\times \arrow[r, hook] \arrow[d] & Q \arrow[d] \\ \mathbb{C}^* \oplus Q^\times \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{C}^* \oplus Q \end{tikzcd}$$

For $f: X \to Y$ a morphism of schemes, $(\mathcal{M}_Y, \alpha_Y)$ a log structure on Y.

$f^*(\mathcal{M}_Y, \alpha)$: look at $f^{-1}\mathcal{M}_Y \xrightarrow{\alpha_Y} f^{-1}\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$ then logify.

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r, "\alpha_Y"] \arrow[rr, bend right, "\alpha"'] & f^{-1}\mathcal{O}_Y \arrow[r, "f^*"] & \mathcal{O}_X \end{tikzcd}$$

"pullback of the log structure"

<u>Example 4</u>: $(\mathbb{A}^n, D = V(x_1, \dots, x_n))$ $\quad p = \{0\} = \operatorname{Spec} \mathbb{C} \hookrightarrow \mathbb{A}^n$

$j^*(\mathcal{M}_D, \alpha) =: M_{p,n}$ "standard n-log structure on the point".

$$\begin{aligned} &\parallel \\ \mathbb{C}^* \oplus \mathbb{N}^n &\xrightarrow{\alpha} \mathbb{C} \\ (a, k_1, \dots, k_n) &\longmapsto a \times 0^{k_1 + \dots + k_n} \end{aligned}$$