Damien - The Kato-Nakayama space

▷ Everything makes sense for $(X, \mathcal{O}_X)$ a locally ringed space.

↳ pre-log structure, log structure

- $(X, \mathcal{O}_X)$ scheme $\rightsquigarrow$ log scheme (Zariski)
- $(X, \mathcal{O}_X)$ analytic space $\rightsquigarrow$ analytic log space

▷ For M a monoid, M is <u>saturated</u> if it is integral and $\left(\begin{array}{c}\forall a \in M^{gp} \\ \forall n \geq 1\end{array} \; a^n \in M \Rightarrow a \in M \right)$

$$\text{integral} \;\updownarrow\; M \hookrightarrow M^{gp}$$

M is <u>fs</u> (fine & saturated) if it is saturated & finitely generated.

▷ "affine" log scheme : M a monoid, $M \longrightarrow \underset{\text{(base ring)}}{\mathbb{K}}[M]$ defines a log scheme denoted by $\mathrm{Spec}(M \to \mathbb{K}[M])$

$X = \mathrm{Spec}(\mathbb{K}[M])$, $\underline{M} \longrightarrow \mathcal{O}_X$ is a pre-log structure, consider the associated log structure "$\underline{M}^a \xrightarrow{\alpha^a} \mathcal{O}_X$".

▷ For $(X, \mathcal{M}_X)$ a log "space" is <u>fs</u> if locally there exists a chart

$(X, \mathcal{M}_X) \xrightarrow{f} \mathrm{Spec}(P \to \mathbb{K}[P])$ with P fs.

such that $\cancel{f^*: (\underline{P})^a \to f^{-1}}$ $\quad f^*\underline{P}^a \xrightarrow{\sim} \mathcal{M}_X$.

From now on we assume that our log "spaces" are fs.

▷ Now take $(X, \mathcal{M}_X)$, define

$$X^{\log} \underset{\text{as a set}}{:=} \left\{ (x, h),\ x \in X(\mathbb{C}),\ h : \mathcal{M}_{X,x}^{(gp)} \longrightarrow S^1 \;\middle|\; \forall f \in \mathcal{O}_{X,x}^{\times},\ h(f) = \frac{f(x)}{|f(x)|} \right\}$$

This is the Kato-Nakayama space (well, for now, set) of $(X, \mathcal{M}_X)$.

~~Recall: the log point~~ Define: $pt = \mathrm{Spec}(\mathbb{C})$ with log structure given by ~~$\mathbb{N}$, or more generally~~ a monoid $T = \mathbb{R}_{\geq 0} \times S^1 \xrightarrow{\alpha} \mathbb{C}$

$$X^{log} \cong \mathrm{Hom}_{\text{log Spaces}}\big((pt, T), (X, \mathcal{M}_X)\big)$$

Indeed, for $x : \mathrm{Spec}(\mathbb{C}) \to X$, we look at morphisms of monoids $\mathcal{M}_{X,x} \to T$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}_{X,x} \arrow[r] \arrow[d, "\alpha"'] & \mathbb{R}_{\geq 0} \times S^1 \arrow[d] \arrow[dr] & \\
\mathcal{O}_{X,x} \arrow[r, "ev_x"'] & \mathbb{C} \arrow[r] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

Now, for $X = \mathrm{Spec}(M \to \mathbb{C}[M])$, a point in X^{log} is a point in X, i.e. $\mathbb{C}[M] \to \mathbb{C}$. We look at:

$$\begin{tikzcd}
 & M \arrow[r] \arrow[dl] \arrow[d] & \mathbb{R}_{\geq 0} \times S^1 \arrow[d] \\
\Gamma(X, \mathcal{O}_X) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathbb{C}
\end{tikzcd}$$

$$X^{log} = \mathrm{Hom}(M, \mathbb{R}_{\geq 0} \times S^1) \simeq \mathrm{Hom}(M, \mathbb{R}_{\geq 0}) \times \mathrm{Hom}(M, S^1)$$

$$\hookrightarrow \mathrm{Hom}(M, \mathbb{C}) \times \mathrm{Hom}(M, S^1) \subset \mathrm{Hom}(M, \mathbb{C}) \times (S^1)^N$$

Take the induced topology on X^{log}, and glue.

In other words, the topology on X^{log} is the weakest so that

1) $X^{log} \xrightarrow{\tau} X(\mathbb{C})$ is continuous, and

2) $\forall m$ local section of $\mathcal{M}_X$ on some open U, $\tau^{-1}(U) \xrightarrow{\arg(m)} S^1$ continuous.
$(x, h) \longmapsto h(m_x)$

<u>Example</u>: $X = \mathbb{A}^1$, with divisorial log structure given by $D = \{0\} \subset X$
$= \mathrm{Spec}(\mathbb{N} \to \mathbb{C}[\mathbb{N}])$

$$X^{\log} = \mathrm{Hom}(\mathbb{N}, \mathbb{T}) \simeq \mathbb{R}_{\geq 0} \times S^1 = \mathrm{Bl}_0(\mathbb{C}) \quad \text{"real oriented blow-up".}$$

Another way to view this:

$$\mathrm{Bl}_0(\mathbb{C}) = \left\{ (z, w) \in \mathbb{C} \times S^1 \;\middle|\; z = |z| w \right\}$$

<u>Example</u>: $(pt, \mathbb{N})^{\log} = S^1$

<u>Example</u>: take $\mathbb{N} \to \mathbb{C}[x]$, $1 \mapsto x^2$, defines a log structure on $\mathbb{A}^1$.

<u>Lemma A</u>: (1) $\tau : X^{\log} \to X$ is continuous and proper

(2) $\forall x \in X$, $\tau^{-1}(x) \simeq (S^1)^r$ $\quad r = \mathrm{rk}(\mathcal{M}^{gp}_{X,x} / \mathcal{O}^\times_{X,x})$.

(3) If $f : X \to Y$ is such that $f^*\mathcal{M}_Y \simeq \mathcal{M}_X$, then $X^{\log} \simeq X \times_Y Y^{\log}$.

<u>Idea of proof</u>:

(1) locally $X^{\log} \hookrightarrow X \times (S^1)^N$.

(2) Clear by (3) by using $x : \mathrm{Spec}(\mathbb{C}) \to X$. $\blacksquare$

(3) locally on X: choose a chart $P \to \mathcal{M}_Y$

$$\begin{tikzcd}
X^{\log} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y^{\log} \arrow[d] \\
X \times \mathrm{Hom}(P, S^1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \times \mathrm{Hom}(P, S^1) \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

Real-oriented blow-ups :

▷ Case of a smooth divisor $D \subset X$ with X smooth.

Locally $D = \{z_1 = 0\} \subset \mathbb{C}^n$

open tubular neighbourhood of D in X.

$$\mathrm{Blo}_D(X) = \{(z_1, \dots, z_n, w) \in \mathbb{C}^n \times S^1 \mid z_1 = |z_1| w\} \overset{\text{homeo}}{\simeq} X \setminus U$$

▷ Case of a normal crossing divisor $D = D_1 \cup \dots \cup D_k$ where locally $D_i = \{z_i = 0\} \subset \mathbb{C}^n$.

$$\mathrm{Blo}'_D(X) := \mathrm{Blo}_{D_1}(X) \times_X \dots \times_X \mathrm{Blo}_{D_k}(X)$$

$$= \{(z_1, \dots, z_n, w_1, \dots, w_k) \in \mathbb{C}^n \times (S^1)^k \mid z_i = |z_i| w_i\}$$

Claim: if $\mathcal{M}$ is the divisorial log structure on X associated to D a ncd, then

$$(X, \mathcal{M})^{\log} \simeq \mathrm{Blo}'_D(X)$$