Clément

De Rham theory of logarithmic varieties

Goal: X log complex-analytic variety

· Define a complex of sheaves $\omega_X^{\cdot}$

· Prove $\mathbb{H}^{\cdot}(X, \omega_X^{\cdot}) \simeq H^{\cdot}_{sing}(X^{log}; \mathbb{C})$

(under some assumptions) next week Kato-Nakayama space

Not in this talk: · Algebraic version

· Relative version

0) Precursors

X complex manifold. Holomorphic Poincaré Lemma:

$$\mathbb{C}_X \xrightarrow{\sim} \Omega_X^{\cdot}$$

$$\Rightarrow \mathbb{H}^{\cdot}(X, \Omega_X^{\cdot}) \simeq H^{\cdot}_{sing}(X; \mathbb{C})$$

X complex manifold $\supset D$ normal crossing divisor

$U = X \setminus D \xhookrightarrow{j} X$

$\Omega_X^{\cdot}(\log D) \subset j_* \Omega_U^{\cdot}$, spanned over $\Omega_X^{\cdot}$ by the $\frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_k}}{z_{i_k}}$

forms with logarithmic singularities

where, in local coordinates, $X = \Delta^n$, $D = \{z_1 \cdots z_n = 0\}$

<u>Fact</u> (Atiyah–Hodge '55, Deligne): $\mathbb{H}^\bullet(X, \Omega_X^\bullet(\log D)) \simeq H^\bullet_{\mathrm{sing}}(U; \mathbb{C})$

<u>Proof</u>: Locally (worst case), $U = (\Delta^\times)^n \subset X = \Delta^n$ with cohomology

$$\begin{tikzcd}
\Lambda^\bullet(\mathbb{C}\, d\log(z_1) \oplus \cdots \oplus \mathbb{C}\, d\log(z_n)) \arrow[d, "\text{natural map}"] \arrow[dd, bend right=60, "\mathrm{id}"'] \\
H^\bullet(\Gamma(X, \Omega_X^\bullet(\log D))) \arrow[d] \\
H^\bullet(\Gamma(X, \Omega_X^\bullet(\log D)) / (z_1, \dots, z_n)) \arrow[d, phantom, "\parallel"] \\
\Lambda^\bullet(\mathbb{C}\, d\log(z_1) \oplus \cdots \oplus \mathbb{C}\, d\log(z_n))
\end{tikzcd}$$

Want to prove: if $\alpha \in \Gamma(X, \Omega_X^\bullet(\log D))$ st $d\alpha = 0$, then α is cohomologous to a log form with constant coefficient.

$\hookrightarrow$ proof by induction on the number of components of D on which α has a pole.

Base case: holomorphic Poincaré lemma.

Induction step: $\alpha = \underset{\substack{\uparrow \\ \text{no pole} \\ \text{at } z_n = 0}}{\beta} + \underset{\substack{\uparrow \\ \text{independent of } z_n}}{\gamma} \wedge \frac{dz_n}{z_n}$

$$d\alpha = d\beta + d\gamma \wedge \frac{dz_n}{z_n} = 0 \implies d\beta = d\gamma = 0$$

by induction hypothesis, $\beta = d\varphi + \beta'$ ← constant coeff

$\gamma = d\psi + \gamma'$ ←

and thus $\alpha = d\left(\varphi + \psi \wedge \frac{dz_n}{z_n}\right) + \underbrace{\beta' + \gamma' \wedge \frac{dz_n}{z_n}}_{\text{constant coeff}}$

1) Definition of $\omega_X^\cdot$

$X = (\underset{\substack{\uparrow \\ \text{complex analytic variety}}}{X}, M_X \xrightarrow{\alpha} \mathcal{O}_X)$ fs

Recall: fs = X is covered by charts $P_X \to \mathcal{O}_X$ where P is a $\begin{cases} \text{finitely generated monoid} \\ \text{integral} \\ \text{saturated} \end{cases}$

Defn: ω_X^1 is the sheaf of $\mathcal{O}_X$-modules

$$\frac{\Omega_X^1 \oplus \left(\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp}\right)}{\forall m \in \mathcal{M}_X,\ \alpha(m) \otimes m = d(\alpha(m))}$$

Notation: $m \in \mathcal{M}_X$ is denoted by $d\log(m) \in \omega_X^1$, so the relation reads $\alpha(m)\, d\log(m) = d\alpha(m)$

If $m \in \alpha^{-1}(\mathcal{O}_X^*) \simeq \mathcal{O}_X^*$ then $d\log(m) = \dfrac{d\alpha(m)}{\alpha(m)}$

In particular, for the trivial log structure, $\omega_X^1 = \Omega_X^1$.

Defn: $\left(\omega_X^\bullet := \Lambda^\bullet(\omega_X^1),\ d\right)$ st $d(d\log(m)) = 0$.

2) Examples

- Divisorial case (normal crossing case)
  locally $\mathcal{M}_X = \mathcal{O}_X^* z_1^{\mathbb{N}} \cdots z_r^{\mathbb{N}} \xrightarrow{\alpha} \mathcal{O}_X$

  $$\omega_X^\bullet = \Omega_X^\bullet(\log D)$$

- Log point $X = pt$, $\mathbb{C} z^{\mathbb{N}} \xrightarrow{\alpha} \mathbb{C}$

$$z \longmapsto 0$$

$$\omega^1_X = \mathbb{C}\, d\log(z) \qquad \omega^\bullet_X : 0 \to \mathbb{C} \xrightarrow{d=0} \mathbb{C}\, d\log(z) \to 0$$

note: it does compute $H^\bullet_{sing}(\underbrace{X^{\log}}_{S^1}; \mathbb{C})$

- More generally, take P an fs monoid, consider

$$X = \mathrm{Spec}\,(P \to \mathbb{C}[P])^{an}$$

then

$$\mathcal{O}_X \otimes_{\mathbb{Z}} P^{gp} \xrightarrow{\sim} \omega^1_X$$
$$p \longmapsto d\log(p)$$

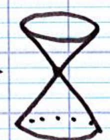
surjectivity follows from: Ω^1_X is $\mathcal{O}_X$-spanned by dp for $p \in P$, and $dp = p\, d\log(p)$.

In particular, ω^1_X is a free $\mathcal{O}_X$-module.

Contrast: Ω^1_X is not always (locally) free

Example: $P = u^{\mathbb{N}} v^{\mathbb{N}} w^{\mathbb{N}} / (w^2 = uv)$

$$\mathbb{C}[P] = \mathbb{C}[u,v,w] / (w^2 - uv)$$

$X =$ 

$$\Omega^1_X = \frac{\mathcal{O}_X du \oplus \mathcal{O}_X dv \oplus \mathcal{O}_X dw}{2w\,dw - u\,dv - v\,du}$$

not locally free near $(0,0,0)$.

$$\omega^1_X = \frac{\mathcal{O}_X d\log(u) \oplus \mathcal{O}_X d\log(v) \oplus \mathcal{O}_X d\log(w)}{2\,d\log(w) - d\log(u) - d\log(v)} \simeq \mathcal{O}_X^2$$

- For $X = \mathrm{Spec}(P \to \mathbb{C}[P])^{an}$, take

$$Z = \mathrm{Spec}(P \to \mathbb{C}[P]/(\Sigma))^{an} \hookrightarrow X$$ where $\Sigma \subset P$ is an ideal

Then $\omega^1_Z \simeq \mathcal{O}_Z \otimes_{\mathbb{Z}} P^{gp}$

3) <u>Theorem</u>: Assume that X is (locally) $\mathrm{Spec}(P \to \mathbb{C}[P]/(\Sigma))$ for P fs and $\Sigma \subset P$ ideal. Then:

$$\Lambda^q((\mathcal{M}_X^{gp}/\mathcal{O}_X^\times) \otimes_{\mathbb{Z}} \mathbb{C}) \xrightarrow{\sim} \mathcal{H}^q(\omega_X^\bullet)$$

$$m \longmapsto [d\log(m)]$$

Sketch of proof:

- $P = z_1^{\mathbb{N}} \cdots z_n^{\mathbb{N}}$ Done at the beginning
- case of a point: $P^\times = \{1\}$, $\mathbb{C}^\times \times P \xrightarrow{\alpha} \mathbb{C}$ $\quad (\Sigma = P \setminus \{1\})$

  $\omega_X^\bullet = \left(\Lambda^\bullet(P^{gp} \otimes_{\mathbb{Z}} \mathbb{C}),\ d=0\right)$ trivially true.

- Worst case scenario: $\Sigma = \emptyset$, $X = \mathrm{Spec}\left(P \to \mathbb{C}[P]\right)$ with P fs, $P^\times = \{1\}$.

  $\Leftrightarrow P = S_\sigma$ for some strongly convex cone σ

Take stalk at x = the distinguished point of X.

$$\underbrace{\mathcal{H}^q(\omega_X^\bullet/m_x)}_{\text{is of the previous form}} \xleftarrow{\ id\ } \Lambda^q\left((\mathcal{M}_X^{gp}/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{C}\right)_x$$

Use toric resolution of singularities to prove that:

$$\left(\mathcal{H}^q(\omega_X^\bullet)\right)_x \hookrightarrow \mathcal{H}^q(\omega_X^\bullet/m_x)$$

$\rightsquigarrow$ reduce to the case of free monoid = 1st case.

use Demazure vanishing:

$$\begin{array}{c} Y \\ {\scriptstyle f}\big\downarrow {\scriptstyle \text{resolution}} \\ X \end{array} \qquad \mathcal{O}_X \xrightarrow{\sim} Rf_*\mathcal{O}_Y$$

then $\omega_X^\bullet \xrightarrow{\sim} (R)f_*\omega_Y$