Intro

$$f(x) = a_0 + a_1 x + \dots + a_m x^m$$

$$g(x) = b_0 + b_1 x + \dots + b_n x^n$$

$\boxed{m, n \text{ degrés fixés}}$

$f, g \longmapsto R(f,g)$ <u>résultant</u> de f et g ; polynôme irréductible en les coeffs a_i, b_j, qui s'annule quand f et g ont une racine commune.

$f \longmapsto \Delta(f)$ <u>discriminant</u> de f : polynôme irréductible en les coeffs a_i, qui s'annule quand f a une racine double

(définitions à constante multiplicative non nulle près)

$$\Delta(f) = R(f, f')$$

<u>Formule de Sylvester</u> :

$$R(f,g) = \det \begin{pmatrix} a_m & a_{m-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_m & \cdots & a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & a_m & \cdots & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & \ddots & & 0 \\ & \ddots & & & \ddots & & \\ 0 & & \ddots & b_n & \cdots & \cdots & b_0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n-1 \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m-1 \end{matrix}$$

$X \subset \mathbb{P}^n$ variété projective

Chap ① X-discriminant $\longrightarrow$ dualité projective

Chap ② Cayley method $\longrightarrow$ complexe de Koszul, déterminant de complexes

Chap ③ X-resultant $\longrightarrow$ grassmannienne, pt de Chow

Chap ④ Chow variety $\longrightarrow$ variété qui paramètre les cycles dans $\mathbb{P}^n$

↑ Partie 1 de GKZ

Partie 2 : le cas où X est torique

1) Dualité projective ~~et~~ discriminant

V complex vector space $\qquad \mathbb{P}(V)$ proj ~~space~~ space of lines in V (1-dim subspaces)

V^* dual vector space $\qquad \mathbb{P}(V)^* = \mathbb{P}(V^*)$ proj space of lines in V^* = hyperplanes in V

~~projective duality~~

$p \in \mathbb{P}(V)$, $p = [x]$	$\longleftrightarrow \; p^\vee = \{[\ell] \,;\, \ell(x) = 0\} \subset \mathbb{P}(V)^*$ = set of hyperplanes containing p; it is a hyperplane in $\mathbb{P}(V^*)$

$X \subset \mathbb{P}(V)$ irreducible alg. subvariety

$H \subset \mathbb{P}(V)$ a hyperplane is <u>tangent to X</u> if $\exists x \in X$ smooth pt st $x \in H$ and the tangent space to H at x contains the tangent space to X at x.

<u>Projective duality</u>: $\quad X^\vee := \overline{\{\text{hyperplane tangent to } X\}} \subset \mathbb{P}(V)^*$ is called the <u>projective dual</u> to X at x

<u>Definition</u>: The <u>X-discriminant</u> Δ_X is the defining polynomial of $X^\vee$ if $X^\vee$ is a hypersurface in $\mathbb{P}(V)^*$, and if codim $X^\vee > 1$, $\Delta_X := 1$

Example: $X \subset \mathbb{P}^d$ the Veronese curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$

$$X = \{ [x^d : x^{d-1}y : x^{d-2}y^2 : \dots : xy^{d-1} : y^d] \ ; \ (x:y] \in \mathbb{P}^1 \}$$

then Δ_X = classical discriminant of a degree d polynomial.

Theorem (biduality thm): $\forall X \subset \mathbb{P}(V)$ projective variety, $(X^\vee)^\vee = X$.
Furthermore, if z is a smooth pt of X, H is a smooth pt of $X^\vee$, then H is tangent to X at z iff z (regarded as a hyperplane in $\mathbb{P}(V)^*$) is tangent to $X^\vee$ at H.

Suggestion: ① talk on projective duality & examples
② Proof of biduality thm
[...] e.g. more examples & properties
Chap1 §5 Katz dimension formula & application
$\downarrow$
for $X^\vee$

② The Cayley method

Goal: generalize Sylvester formula:

Step 1: Interpret the vanishing of $\Delta_X(f)$ as a violation of the exactness of a certain complex of coherent sheaves on X

Step 2: Interpret the non exactness of a complex of sheaves as the non exactness of a certain complex of vector spaces whose terms are fixed, and differential varies with f

Step 3: the X-discriminant is the determinant of this complex

"The proof is slightly more technical than the previous material and makes use of the formalism of derived categories."

Aplication: degree & dimension of $X^\vee$ expressed from the Hilbert polynomial of X.

Suggestions: ③ Setup 1: jet bundles, Koszul complex, discriminantal complex

④ Setup 2: determinant of complex

⑤ Proof

⑥ Aplications

[...] eg Chapter 2, §5 the discriminant as the determinant of a spectral sequence

(5) Chow point & X-resultant

$X \subset \mathbb{P}^{n-1}$ irred subvar of dimension $k-1$

$\mathcal{Z}(X) :=$ set of all $(n-k-1)$-dim projective subspaces L in $\mathbb{P}^{n-1}$ that intersect X.

Proposition: This is an irreducible hypersurface of degree d in the grassmannian $G(n-k,n)$.

(of $(n-k-1)$-dim subspaces in $\mathbb{P}^{n-1}$)

degree d? grassmannian is Picard rk 1 $\Rightarrow$ there is a natural projective embedding

$G(n-k,n) \subset \operatorname{Proj} \mathcal{B}$

$\mathcal{B} = \bigoplus_{m \geq 0} \mathcal{B}_m$ total coordinate ring of the Grassmannian

$\mathcal{Z}(X)$ is defined by the vanishing of some element $R_X \in \mathcal{B}_d$.

Definition: R_X is called the ~~Chow form~~ Chow form or X-resultant of X, defined up to nonzero multiple.

$[R_X] \in \mathbb{P}(\mathcal{B}_d)$ is called the Chow point of X.

Example: $\mathbb{P}^1 \hookrightarrow \mathbb{P}(S^d\mathbb{C}^2) = \mathbb{P}^{d+1}$ Veronese embedding again

R_X suitably interpreted coincides with the classical resultant of 2 ~~homogeneous~~ 1-variable polynomials of degree d. ~~tbs~~

Theorem (Cayley trick): $\tilde{X} := X \times \mathbb{P}^{k-1}$

$$\underset{X\text{- resultant}}{R_X} = \underset{\tilde{X}\text{- discriminant}}{\Delta_{\tilde{X}}}$$

$\Rightarrow$ can apply the Cayley method to X-resultants as well.

~~Example: $R(f(x), g(y)) = \Delta(y f(x) + x g(y))$~~

Suggestions:
⑦ Geometry of Grassmannians
⑧ ~~Chow~~ X-resultant
⑨ Cayley trick & application of Cayley method
[...] e.g. Chapter 3 §3 Mixed resultants

(4) Chow varieties

Defn: cycle of dimension $k-1$ in $\mathbb{P}^{n-1}$:

$$X = \sum_i m_i X_i$$

formal finite linear ~~combination~~ combination with non-negative integer coeff of irred subvar of dim $k-1$ in $\mathbb{P}^{n-1}$.

$\deg(X) := \cancel{\prod_i (\deg X_i)^{m_i}}$ $\sum m_i \deg(X_i)$

Define $R_X := \prod_i R_{X_i}^{m_i} \in \mathcal{B}_{\deg(X)}$

Theorem [Chow-Van der Waerden]: The map $X \mapsto R_X$ defines an embedding of $G(k,d,n)$ = set of all $(k-1)$-dim cycles in $\mathbb{P}^{n-1}$ of degree d as a closed algebraic variety in $\mathbb{P}(\mathcal{B}_d)$.

Suggestions: (10) Proof of this thm

[...] Chapter 4 §2 : $G(1,d,n)$ variety of 0-cycles

§3 : Cayley-Green-Morrison equations of Chow varieties.