GT GKZ Sylvain (15/04/26) : supplement

We give here a proof of the following "<u>Key point / proposition</u>" :
$$\begin{cases} R{p_2}_*(p_1^*\mathcal{M}) \simeq \mathcal{O}(C_-) \\ R{p_2}_*(p_1^*\mathcal{M} \otimes \mathcal{E})[-r] \simeq \mathcal{O}(C_+) \end{cases}$$
where $\mathcal{E} = i^*\left(p_X^*\Lambda^r J(\mathcal{E}) \otimes p_V^*\mathcal{O}_{\mathbb{P}(V)}(r)\right)$
(the proof was skipped during the talk).

<u>Notations</u>

$$\begin{tikzcd}
 & & W \arrow[lldd, bend right, "p_1"'] \arrow[d, hook, "i"] \arrow[rr, "A"] \arrow[rdd, "p_2"] & & X^\vee \arrow[ldd, hook, "j"] \\
 & & X \times \mathbb{P}(V) \arrow[lld, "p_X"'] \arrow[rd, "p_V"] & & \\
X & & & \mathbb{P}(V) &
\end{tikzcd}$$

<u>Reminder 1</u> (projection formula) : Let $f : X \to Y$ be a scheme morphism. Then $\forall \mathcal{F} \in (\mathcal{O}_X\text{-Mod})$, $\forall \mathcal{E} \in (\mathcal{O}_Y\text{-Mod})$ with $\mathcal{E}$ <u>locally free of finite rank</u>,
$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq (R^i f_*\mathcal{F}) \otimes \mathcal{E} \qquad \forall i \geq 0$$

<u>Reminder 2</u> (flat base change) : Let
$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$
be a cartesian square of schemes, with <u>f quasi-compact</u> and <u>u flat</u>. Then $\forall \mathcal{F} \in QCoh(X)$, $\forall i \geq 0$, $(R^i f'_*) v^*\mathcal{F} \simeq u^* R^i f_*\mathcal{F}$.

<u>Homological Lemma</u> : Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough injectives, and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Let $A^\bullet \in D^b(\mathcal{A})$ be a complex such that
$$\forall j, \ \forall i > 0, \quad (R^i F)(A^j) = 0$$
(ie the pieces of $A^\bullet$ are F-acyclic)
Then, in $D^b(\mathcal{B})$, $(RF)(A^\bullet) \simeq \underbrace{FA^\bullet}_{\text{apply } F \text{ term by term}}$.

<u>Proof of the homological lemma</u>

Let $A^\bullet \to I_A^\bullet$ be an injective resolution of $A^\bullet$.

Applying F term by term, we get a morphism

$$FA^\bullet \longrightarrow F(I_A^\bullet) =: (RF)(A^\bullet)$$

We prove that this is a quasi-isomorphism (ie an isom in $D^b(\mathcal{B})$).

As for any derived functor, there is a spectral sequence:

$$E_1^{pq} = (R^qF)(A^p) \Longrightarrow H^{p+q}(RF(A^\bullet))$$

By assumption, $E_1^{pq} = 0 \quad \forall q > 0, \forall p$.

Hence the sequence degenerates to isomorphisms

$$H^n(FA^\bullet) \xrightarrow{\sim} H^n(RF(A^\bullet)) \quad \forall n.$$ ∎

Proof of the proposition

$$Rp_{2*}(p_1^*\Omega \otimes \mathcal{E}) = R(p_V \circ i)_*\left((i^* p_X^*\Omega) \otimes \mathcal{E}\right)$$

$$= (Rp_{V*})\, i_*\left((i^* p_X^*\Omega) \otimes \mathcal{E}\right) \qquad [Ri_* = i_* \text{ because } i \text{ is affine}]$$

$$\simeq (Rp_{V*})\left((p_X^*\Omega) \otimes i_*\mathcal{E}\right) \qquad \text{by the projec° formula}$$

($p_X^*\Omega$ is loc. free of rk 1)

Let $\mathcal{K} = K_+(E, \sigma)$ (see the notes of the talk).

We have seen that $\mathcal{K}$ is a resolution of $i_*\mathcal{E}$ (placed in degree r), hence $i_*\mathcal{E} \simeq \mathcal{K}[r]$ in $D^b(\mathrm{Coh}(X \times \mathbb{P}(V)))$

We get $Rp_{2*}(p_1^*\Omega \otimes \mathcal{E}) \simeq Rp_{V*}(p_X^*\Omega \otimes \mathcal{K}[r]) = Rp_{V*}(p_X^*\Omega \otimes \mathcal{K})[r]$

- We apply the homological lemma to the complex $p_X^*\Omega \otimes \mathcal{K}$.

Its degree j component is by def°

$$p_X^* \Lambda^j J(\mathcal{L}) \otimes p_V^* \mathcal{O}(j) \otimes p_X^* \Omega$$

Let us prove that it is p_V^*-acyclic:

$$\forall i \geq 0 \quad R^i p_{V*}\left(p_X^*(\Lambda^j J(\mathcal{L}) \otimes \Omega) \otimes p_V^*\mathcal{O}(j)\right)$$

$$\simeq R^i p_{V*}\left(p_X^*(\Lambda^j J(\mathcal{L}) \otimes \Omega)\right) \otimes \mathcal{O}(j) \qquad [\text{projec° formula}]$$

$$\simeq u^* H^i(X, \Lambda^j J(\mathcal{L}) \otimes \Omega) \otimes \mathcal{O}(j)$$

by base change in the square

$$\begin{tikzcd} X \times \mathbb{P}(V) \arrow[r, "p_X"] \arrow[d, "p_V"'] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \mathbb{P}(V) \arrow[r, "u"'] & \mathrm{Spec}\, k \end{tikzcd}$$

$$= \begin{cases} 0 & \text{if } i > 0 \text{ (by choice of } \Omega) \\ \mathcal{O}(C_+)^j & \text{if } i = 0 \text{ (by def of } \mathcal{O}(C_+)) \end{cases}$$

Hence we can apply the lemma, and we see that

$$(Rp_{V*})(p_X^*\Omega \otimes \mathcal{K}) \simeq \mathcal{O}(C_+).$$

The proof for $Rp_{2*}(p_1^*\Omega)$ is similar, replacing $\mathcal{E}$ with $\mathcal{O}_W$ and $K_+(E, \sigma)$ with $K_-(E, \sigma)$. ∎