$\boxed{\text{GTGK Z}}$ Ulysse 20/05/26

$G(k,n) = \{(k-1)\text{-dim proj subspaces of } \mathbb{P}^{n-1}\}$

$= \mathrm{Proj}\left(\bigoplus_{d\geq 0} B_d\right)$

$B_\bullet = \mathbb{C}[p_I, \ I \subset [\![1,n]\!], \ \#I = k] \big/ \langle \text{Plücker relations} \rangle$

Line bundle $\mathcal{O}_{G(k,n)}(d)$ st $B_d = \Gamma(\mathcal{O}_{G(k,n)}(d))$ and $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$

$\forall$ flag $N \subset M \subset \mathbb{P}^{n-1}$ with $\dim N = k-2$, $\dim M = k$

$P_{NM} = \{W \in G(k,n) \,;\, N \subset W \subset M\} \simeq \mathbb{P}^1$

If $Z \subset G(k,n)$ hypersurface, $\deg(Z) = \# Z \cap P_{NM}$ for generic $N \subset M$

$$\{\text{hypersurface } Z \subset G(k,n)\} \xleftrightarrow{1:1} \bigsqcup_{d>0} \mathbb{P}(B_d)$$
$$V(f) \longleftarrow\!\!\mapsfrom f$$

<u>Claim</u> : $\deg(V(f)) = \deg f$

Let $f \in B_d \setminus \{0\}$, $N \subset M$ generic, WTS $\#(V(f) \cap P_{NM}) = d$

$\Leftrightarrow \# V(f|_{P_{NM}}) = d$

It suffices to show $\mathcal{O}_{G(k,n)}(d)\big|_{P_{NM}} \simeq \mathcal{O}_{P_{NM}}(d)$

Because $\mathcal{O}(d) \simeq \mathcal{O}(1)^{\otimes d}$, it suffices to treat the case $d=1$, for one particular section f.

Can assume $f = P_{\{1,\dots,k\}}$

$$V(f) = \{W \in G(k,n) \mid W \cap \langle e_{k+1}, \dots, e_n \rangle \neq \emptyset$$

$$V(f) \cap P_{NM} = \{W \in G(k,n) \; ; \; N \subset W \subset M \, , \; W \cap \langle e_{k+1}, \dots, e_n \rangle \neq \emptyset\}$$

$$= \{\langle N, M \cap \langle e_{k+1}, \dots, e_n \rangle \rangle\}$$

$$\#(\qquad) = 1$$

2) Associated hypersurface

$X \subset \mathbb{P}^{n-1}$ irreducible, $\dim X = k-1$, $\deg X = d$

$Z(X) := \{W \in G(n-k,n), \; W \cap X \neq \emptyset\}$ associated hypersurface
as a set for now

$B(X) = \{(x, W) \in X \times G(n-k,n) \; ; \; x \in W\}$ subvariety of $X \times G(n-k,n)$

$$\begin{tikzcd} & & B(X) \arrow[dll] \arrow[dl] \arrow[dr] & \\ Z(X) & \subset G(n-k,n) & & X \end{tikzcd}$$

$Z(X)$ is the image of $B(X)$ in $G(n-k,n)$

Prop: $\mathcal{Z}(X)$ is an irreducible hypersurface of degree d.

Proof:
$$\begin{array}{c} \mathcal{B}(X) \\ \downarrow \\ X \end{array} \quad \text{is a } G(n-k-1, n-1) \text{ fibration}$$

$$\left(\mathcal{B}(X)_x = \{ W \in G(n-k, n) , x \in W \} \right)$$

$\Rightarrow \mathcal{B}(X)$ irreducible

$\Rightarrow \mathcal{Z}(X)$ irreducible

Moreover, $\dim \mathcal{B}(X) = \dim X + \dim (G(n-k-1, n-1))$
$$= k(n-k) - 1 = \dim(G(\cancel{k}_{n-k}, n)) - 1$$

Claim:
$$\begin{array}{c} \mathcal{B}(X) \\ \downarrow \\ \mathcal{Z}(X) \end{array} \quad \text{is birational}$$

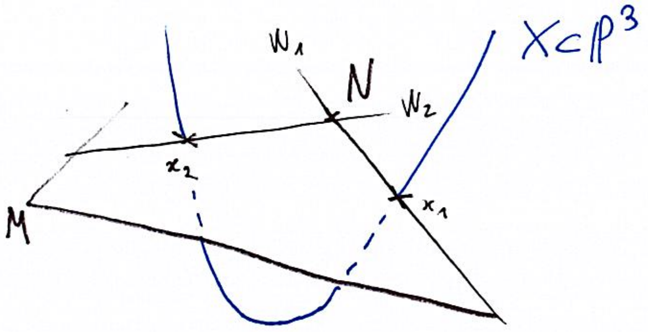
$$\dim \Big(\mathcal{B}(X) \times_{\mathcal{Z}(X)} \mathcal{B}(X) \setminus \overset{\overset{\text{diagonal}}{\downarrow}}{\mathcal{B}(X)} \Big) = \dim (X \times X) + \dim (G(n-k-2, n-2))$$
$$= k(n-k) - 2$$

$$\| $$

$$\{ (x, x', W) , \ x \neq x' , \ x, x' \in W \}$$

$\Rightarrow \dim \mathcal{Z}(X) = \dim \mathcal{B}(X) = \dim (G(n-k, n)) - 1$

$\Rightarrow \mathcal{Z}(X)$ is an irreducible hypersurface.



$$P_{NM} \cap Z(X) = \{ N \subset W \subset M ;\ W \cap X \neq \emptyset \}$$

$$= \{ \langle N, x \rangle ,\ x \in X \cap M \} \qquad \# X \cap M = d \text{ for } M \text{ generic}$$

$$= \{ \langle N, x_1 \rangle, \dots, \langle N, x_d \rangle \}$$

$$= \{ W_1, \dots, W_d \} \qquad \text{all distinct for } N \text{ generic}$$

$\square$

Defn: The X-resultant $R_X \in \mathbb{P}(\mathcal{B}_d)$ is st $Z(X) = V(R_X)$.

Recall: $W \in G(n-k, n)$ is given by k linear equations, well defined up to the action of $GL(k)$. $\qquad M = (v_{ij}) = \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}$

$$\mathcal{B}_\bullet \subset \mathbb{C}[v_{ij}] \qquad \begin{matrix} 1 \leq i \leq k \\ 1 \leq j \leq n \end{matrix}$$

$$\mathcal{B}_d = \{ P \in \mathbb{C}[v_{ij}] ;\ P(gM) = \det(g)^d P(M) \}$$

Let $\tilde{R}_X \in \mathbb{C}[v_{ij}]$ be the image of R_X

$$\tilde{R}_X(f_1, \dots, f_k) = 0 \iff \begin{cases} f_1 = 0 \\ \vdots \\ f_k = 0 \end{cases} \text{has a solution on } X.$$

$\rightarrow$ explains the name X-resultant.

Ex: $X = \mathbb{P}(V) \hookrightarrow \mathbb{P}(S^d V)$ $\dim V = k$

$f_1, \dots, f_k \in (S^d V)^*$ $f_i|_X$ is a homogeneous polynomial of degree d

$\tilde{R}_X(f_1, \dots, f_k) = 0$ iff $\exists x \in X,\ f_1(x) = \dots = f_k(x) = 0$

classical resultant

Rem: X can be recovered from $Z(X)$:

$x \in X$ iff $\forall W \in G(n-k, n),\ x \in W \Rightarrow W \in Z(X)$

Clément

Sylvester's formula

$\mathbb{P}^1 = \mathbb{P}((\mathbb{C}x \oplus \mathbb{C}y)^*)$

$\mathbb{P}^d = \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}x \oplus \mathbb{C}y)^*)$

$$X = \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$$
$$(x:y) \longmapsto (x^d : x^{d-1}y : \dots : y^d)$$

$$X^\vee = \{[f] \in \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}x \oplus \mathbb{C}y)) \mid \exists a \in X,\ T_a X \subset [f]\}$$

$$= \{[f] \in \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}x \oplus \mathbb{C}y)) \mid f \text{ has a root of multiplicity} \geq 2\}$$

$$= \{\Delta_X(f) = 0\}$$

Recall: jet bundle $J(\mathcal{O}_{\mathbb{P}^d}(1)|_X) \ni j(f)$

$[f] \in X^\vee \Leftrightarrow j(f)$ vanishes at some point $a \in X$

$\Leftrightarrow K_-(J(\mathcal{O}_{\mathbb{P}^d}(1)|_X), j(f))$ is not exact.

Here, $\mathcal{O}_{\mathbb{P}^d}(1)|_X = \mathcal{O}_{\mathbb{P}^1}(d)$

$$J(\mathcal{O}(d)) \simeq \mathcal{O}(d-1) \oplus \mathcal{O}(d-1)$$
$$j(f) \longleftrightarrow (\partial_x f, \partial_y f)$$

K_-:

$$0 \to \mathcal{O}(2-2d) \xrightarrow[\begin{pmatrix} -\partial_y f \\ \partial_x f \end{pmatrix}]{} \mathcal{O}(1-d) \oplus \mathcal{O}(1-d) \xrightarrow{(\partial_x f, \partial_y f)} \mathcal{O}_X \to 0$$

Recall: on $\mathbb{P}^1$, for $k \geq -1$, $H^i(\mathbb{P}^1, \mathcal{O}(k)) = 0$ if $i > 0$

$$H^0(\mathbb{P}^1, \mathcal{O}(k)) = \mathrm{Sym}^k(\mathbb{C}x \oplus \mathbb{C}y)$$

Twist K_- by $\mathcal{O}(2d-3)$:

$$0 \to \mathcal{O}(-1) \longrightarrow \mathcal{O}(d-2) \oplus \mathcal{O}(d-2) \longrightarrow \mathcal{O}(2d-3) \longrightarrow 0$$

Take global sections

$$0 \to 0 \longrightarrow \mathrm{Sym}^{d-2}(\mathbb{C}x \oplus \mathbb{C}y) \oplus \mathrm{Sym}^{d-2}(\mathbb{C}x \oplus \mathbb{C}y) \xrightarrow{\begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}} \mathrm{Sym}^{2d-3}(\mathbb{C}x \oplus \mathbb{C}y) \to 0$$

Cayley's formula: $\Delta_X(f) = \det(\text{this map})$

write $f = a_0 x^d + a_1 x^{d-1} y + \dots + a_d y^d$

$$= \begin{vmatrix} da_0 & (d-1)a_1 & \cdots & 2a_{d-2} & a_{d-1} & & 0 \\ 0 & \ddots & & & & \ddots & \\ & & da_0 & \cdots & & \cdots & a_{d-1} \\ a_1 & 2a_2 & \cdots & \cdots & da_d & & \\ & \ddots & & & & \ddots & 0 \\ 0 & & \ddots & & & & \\ & & a_1 & \cdots & & \cdots & da_d \end{vmatrix}$$

Example: $d=2$ $\quad f = ax^2 + bxy + cy^2$

$$\begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix} = 4ac - b^2$$

$d=3$ $\quad f = ax^3 + bx^2y + cxy^2 + dy^3$

$$\begin{vmatrix} 3a & 2b & c & 0 \\ 0 & 3a & 2b & c \\ b & 2c & 3d & 0 \\ 0 & b & 2c & 3d \end{vmatrix} = -3(b^2c^2 - 47a^2b^2 - 4ac^3 - 4b^3d + 18abcd)$$