K field (char $\neq 2$)

V n-dim vector space

k integer $0 \leqslant k \leqslant n$

$G(k,V)$ Grassmannian of k-dim subspaces of V.

Examples:

$k=1 \rightsquigarrow \mathbb{P}(V)$

$k=n-1 \rightsquigarrow \mathbb{P}(V^*)$

First interesting case: $G(2,V)$ where $\dim V = 4$

1) Various considerations

a) Universal property

Actually we should construct a couple $(G, \mathcal{U})$ st:

- G scheme$/K$
- $\mathcal{U}$ locally free subsheaf of $\mathcal{O}_G \times V$ of rank k, st $\mathcal{O}_G \times V / \mathcal{U}$ is also locally free
- for any scheme X and $\mathcal{L}$ ~~sub~~ loc. free subsheaf of $\mathcal{O}_X \times V$ of rank k, st $\mathcal{O}_X \times V / \mathcal{L}$ is loc. free, there is a unique map $f : X \longrightarrow G$ st $\mathcal{L} = f^* \mathcal{U}$.

Rem: i) $\leftrightarrow$ representable functor

ii) in this talk, we will not construct $\mathcal{U}$ and check the universal prop.

b) Coordinates

What are "coordinates" for a k-plane in V?

Example: $G(1,n) \simeq \mathbb{P}(V)$ $\quad$ V with basis $(e_1,\dots,e_n)$

possibilities: i) affine charts: fix e.g. $x_1 = 1$

get coordinates $(x_2, x_3, \dots, x_n)$

ii) consider $(x_1,\dots,x_n) \in V$ as set of coordinates up to the action of $\mathbb{K}^* = GL(1)$

"Stiefel coordinates"

iii) projective coordinates $(x_1 : \dots : x_n)$ (here it is the same)

"Plücker coordinates"

$\boxed{\text{outline}}$

1) done
2) Stiefel coordinates
3) affine coordinates
4) Plücker morphism
5) Plücker relations
6) Homogeneous coordinate ring

2) Stiefel coordinates

Fix a basis $(e_1, \dots, e_n)$ of V

$G(k,n) = G(k,V)$

Let $W \subset V$ be a k-plane. To define W, can chose a basis of W,

$w = (w_1, \dots, w_k)$ Rem: when $K = \mathbb{R}$ or $\mathbb{C}$, could use orthonormal bases.

Put the coordinates of w in a matrix $A(w)$ (in rows)

$$
k \updownarrow \underbrace{\begin{bmatrix} w_{11} & \cdots & w_{1n} \\ & & \\ w_{k1} & \cdots & w_{kn} \end{bmatrix}}_{n} \quad \text{of maximal rank } k.
$$

<u>Definition</u>: The <u>Stiefel variety</u> $S(k,n)$ is the set of maximal rank matrices of $\mathcal{M}_{k,n}$.

Any $A \in S(k,n)$ is a set of <u>Stiefel coordinates</u> for the corresponding k-plane (generated by the rows of A).

Different choice of basis w' of $W \Rightarrow$ there is a matrix $P \in GL(k)$ st $A(w) = PA(w')$.

$$G(k,n) = S(k,n) / GL(k)$$

3) Affine coordinates

Notation: for $I=(i_1,\dots,i_k)$ st $1\leq i_1<\dots<i_k\leq n$ and $A\in\mathcal{M}_{k,n}$, A_I is the submatrix obtained by keeping only columns $i_1,i_2,\dots,i_k$.

$$S(k,n)=\bigcup_I\{A\in\mathcal{M}_{k,n}\,;\ \det A_I\neq 0\}$$

Fix I. Put $U_I:=\mathrm{Vect}(e_{i_1},\dots,e_{i_k})$ so that $V=U_I\oplus U_{\bar I}$

$U_{\bar I}:=\mathrm{Vect}(e_j\ ;\ j\notin I)$

Fact: Let W be a k-plane, w a basis of W. Then

$A_I(w)$ is invertible $\iff$ $V=W\oplus U_{\bar I}$

matrix of the projection $W\longrightarrow U_I$ along the decomposition $U_I\oplus U_{\bar I}$ in the bases w and $(e_{i_1},\dots,e_{i_k})$

This enables us to define an isomorphism:

$\{A\in S(k,n)\,;\ \det(A_I)\neq 0\}=:S_I$

$$S_I/GL(k)\xrightarrow{f_I}\mathcal{M}_{k,n-k}$$
$$A\longmapsto\left(A_I^{-1}A\right)_{\bar I}$$

Transition maps are algebraic.

Example in $G(2,4)$: $I=(1,2)$ $J=(2,3)$

$$\begin{tikzcd}
 & \mathcal{M}_{2,2} \arrow[d] & \begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix} \arrow[d, mapsto] \\
S_I\cap S_J/GL(2) \arrow[ur, "f_I"] \arrow[r, "f_J"'] & \mathcal{M}_{2,2} & \frac{-1}{m_{11}}\begin{bmatrix} m_{21} & m_{21}m_{12}\\ 1 & m_{22}\end{bmatrix}
\end{tikzcd}$$

$$\left[\begin{array}{cc|c|c} 1 & 0 & m_{11} & m_{12}\\ 0 & 1 & m_{21} & m_{22}\end{array}\right]$$

$\det\neq 0$
$\Updownarrow$
$m_{11}\neq 0$

$G(k,n)$ is separated

Proof: Let W, W' be distinct k-planes. We have to prove that W, W' are in a common affine chart. Let $\bar{U}$ be a common supplementary space of W and W', get an affine chart

$$\left\{ \begin{array}{l} K \text{ } k\text{-planes st} \\ K \oplus \bar{U} = V \end{array} \right\} \longrightarrow \mathrm{Hom}(W, \bar{U})$$

$$\text{graph of } f \longleftarrow f$$

common affine chart for W and W'.

4) Plücker morphism

a) Exterior algebra

$$\Lambda^\bullet V = (K \oplus V \oplus V \otimes V \oplus \dots) / \langle x \otimes x, x \in V \rangle$$

$$= K \oplus V \oplus \Lambda^2 V \oplus \dots$$

A basis for $\Lambda^k V$ is given by $e_{i_1} \wedge \dots \wedge e_{i_k}$ $\quad (i_1, \dots, i_k) \subset [\![1, n]\!]$, $i_1 < i_2 < \dots < i_k$

$$\dim \Lambda^k V = \binom{n}{k}$$

Let W be a k-plane, and $(w_1, \dots, w_k)$ a basis of W.

For any automorphism $g : W \to W$, we have

$$g(w_1) \wedge \dots \wedge g(w_k) = \underbrace{\det(g)}_{\neq 0} w_1 \wedge \dots \wedge w_k$$

This gives a well defined map

$$p : G(k,V) \longrightarrow \mathbb{P}(\Lambda^k V) \qquad \text{"Plücker morphism"}$$

$$W = \mathrm{Vect}(w_1, \dots, w_k) \longmapsto w_1 \wedge \dots \wedge w_k$$

More precisely, let $A = \begin{bmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & & \vdots \\ w_{k1} & \cdots & w_{kn} \end{bmatrix}$ a Stiefel matrix for W.

Then $p(W) = (w_{11}e_1 + \dots + w_{1n}e_n) \wedge \dots \wedge (w_{k1}e_1 + \dots + w_{kn}e_n)$

$$= \sum_{\substack{I \subset [\![1,n]\!] \\ \#I = k}} \det(A_I)\, e_I$$

$\Rightarrow$ algebraic

Get a morphism of schemes of finite type over K.

Question: What is the image of p?

5) Plücker relations

a) Decomposable k-vectors

vectors in $\Lambda^k V$

Defn: A k-vector x in $\Lambda^k V$ is decomposable if there exists $v_1, \dots, v_k$ st $x = v_1 \wedge \dots \wedge v_k$.

Not easy to check!

Examples in $\Lambda^2 K^4$: $x = e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3$ is decomposable

$x = e_1 \wedge e_2 + e_3 \wedge e_4$ is not

The image of p is the set of decomposable k-vectors.

b) Contraction and duality

Let V^* be the dual of V. For any $k, r \in \mathbb{N}$, and $\varphi \in \Lambda^r V^*$, we have a map :

$$\iota_\varphi : \Lambda^k V \longrightarrow \Lambda^{k-r} V$$

verifying axioms ... $*$ if $\varphi \in V^*$, $x \in V$, $\iota_\varphi(x) = \varphi(x) = \langle \varphi, x \rangle$ (duality pairing)

$* \ \iota_{\varphi \wedge \psi} = \iota_\varphi \circ \iota_\psi$

The contraction can also be defined in the following way:

$$\forall \psi \in \Lambda^{k-r} V^*, \quad \langle \psi, \iota_\varphi(x) \rangle = \langle \varphi \wedge \psi, x \rangle$$

Let $x \in \Lambda^k V$. We will investigate subspaces W st $x \in \Lambda^k W$

Proposition: $x \in \Lambda^k W$ iff $\forall \varphi \in \Lambda^{k-1} V^*$, $\iota_\varphi(x) \in W$

Proof: Let U be a supplementary space of W. We have $V = W \oplus U$,

$$\Lambda^k V = \Lambda^k U \oplus \Lambda^{k-1} U \otimes W \oplus \cdots \oplus \Lambda^k W$$

$$\Lambda^k V^* = \underbrace{\Lambda^k U^* \oplus \Lambda^{k-1} U^* \otimes W^* \oplus \cdots}_{(\Lambda^k W)^\perp \text{ in } \Lambda^k V^*} \oplus \Lambda^k W^*$$

in other words, $(\Lambda^k W)^\perp$ is generated by $\lambda \wedge \varphi$, $\lambda \in W^\perp$, $\varphi \in \Lambda^{k-1} V^*$

$$x \in \Lambda^k W \iff \forall \psi \in (\Lambda^k W)^\perp, \ \langle \psi, x \rangle = 0$$

$$\iff \forall \lambda \in W^\perp, \forall \varphi \in \Lambda^{k-1} V^*, \ \langle \varphi \wedge \lambda, x \rangle = 0$$

$$\iff \forall \lambda \in W^{\perp}, \forall \varphi \in \Lambda^{k-1} V^*, \ \langle \lambda, \iota_\varphi(x) \rangle = 0$$

$$\iff \forall \varphi \in \Lambda^{k-1} V^*, \ \iota_\varphi(x) \in W.$$

$\square$

Put $W_x := \{ \iota_\varphi(x) ; \varphi \in \Lambda^{k-1} V^* \}$, then W_x is the minimal subspace such that $x \in \Lambda^k W_x$.