GT GKZ 13/05

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$\boxed{\text{Recall:}}$

K field of characteristic $\neq 2$

V n-dim K-vector space

$(e_1, \dots, e_n)$ basis of V

$1 \leq k \leq n-1$

$G(k,V) = G(k,n)$ Grassmannian

$S(k,n) = \{ A \in M_{k,n} \; ; \; \mathrm{rk}\, A = k \}$ Stiefel variety

$G(k,n) = S(k,n) / GL_k$

Plücker embedding

$$p : G(k,V) \longrightarrow \mathbb{P}(\Lambda^k V)$$
$$W = \mathrm{Vect}(w_1, \dots, w_k) \longmapsto \overline{w_1 \wedge \dots \wedge w_k}$$

Image of p? set of decomposable k-vectors

~~Proposition~~ Let $x \in \Lambda^k V$

Define $W_x = \{ \iota_\varphi(x) \, , \; \varphi \in \Lambda^{k-1} V^* \}$

contraction by $\varphi \in \Lambda^{k-1} V^* \simeq (\Lambda^{k-1} V)^*$

subspace of V

<u>Prop</u>: Let U be a subspace of V. Then $x \in \Lambda^k U$ iff $W_x \subset U$.

Corollary: $x \overset{\neq 0}{} $ is decomposable iff $\dim W_x = k$

Proof: * if $x = v_1 \wedge \dots \wedge v_k$ decomposable, put $U = \mathrm{Vect}(v_1, \dots, v_k)$.

Since $x \in \Lambda^k U$, $W_x \subset U$ and $\dim W_x \leq k = \dim U$.

If $\dim W_x < k$, then $x \in \Lambda^k W_x = 0$ impossible.

* If $\dim W_x = k$, write $W_x = \mathrm{Vect}(v_1, \dots, v_k)$. Then $\Lambda^k W_x = K v_1 \wedge \dots \wedge v_k \ni x$

hence x is decomposable. □

If $\dim W_x = k$, then for any $w \in W_x$, $w \wedge x = 0$

If $\dim W_x > k$, then there exists $w \in W_x$ st $w \wedge x \neq 0$

Conclusion: $x \in \Lambda^k V$ is decomposable iff $\forall \varphi \in \Lambda^{k-1} V^*$, $\iota_\varphi(x) \wedge x = 0$

In coordinates.

For any k-tuple $I = (i_1, \dots, i_k) \in [\![1,n]\!]^k$ (~~not ordered~~ distinct, ~~increasing~~), let $\sigma(I)$ be the ~~ordered list~~

same list of integers, ordered.

$\sigma(I)$ the permutation in $\mathfrak{S}_k$ sending I to $\sigma(I)$.

$\varepsilon(I) = \varepsilon(\sigma(I))$ the signature.

$e_I = e_{i_1} \wedge \dots \wedge e_{i_k}$ st $e_I = \varepsilon(I)\, e_{\sigma(I)}$

$e_I^* = e_{i_1}^* \wedge \dots \wedge e_{i_k}^*$

The e_I for $I \subset [\![1,n]\!]$, $\#I = k$ form a basis for $\Lambda^k V$

If $x = \sum_{\substack{I \subset [\![1,n]\!] \\ \#I = k}} p_I e_I \in \Lambda^k V$

For any $J = (j_1, \dots, j_k)$, put $p_J = \varepsilon(J) p_{\sigma(J)}$

x is decomposable $\iff \forall I \subset [\![1,n]\!]$, $\#I = k-1$,

$$\iota_{e_I^*}(x) \wedge x = 0$$

Write $\iota_{e_I^*}(x) \wedge x = \sum_{\substack{J \subset [\![1,n]\!] \\ \#J = k+1}} \alpha_{I,J}\, e_J$

$$\boxed{x \text{ is decomposable} \iff \forall I, J \quad \#I = k-1,\ \#J = k+1,\ \alpha_{I,J} = 0}$$

Lemma 1: $\#I = k-1 \qquad \#S = k$

$$\iota_{e_I^*}(e_S) = \begin{cases} 0 & \text{if } I \subsetneq S \\ \varepsilon(I,s) e_s & \text{if } S \setminus I = \{s\} \end{cases}$$

Proof: write $\iota_{e_I^*}(e_S) = \sum a_s e_s$

compute $\langle e_\ell^*, \iota_{e_I^*}(e_S) \rangle$ using duality and contractions. $\square$

Lemma 2: If $x = \sum_{\#S = k} p_S e_S$, $\iota_{e_I^*}(x) = \sum_{s \in [\![1,n]\!] \setminus I} p_{(I,s)} e_s$

Proposition: Let $I = (i_1, \dots, i_{k-1}) \subset [\![1,n]\!]$, $J = (j_1, \dots, j_{k+1}) \subset [\![1,n]\!]$, then

$$\alpha_{I,J} = \sum_{s \in J \setminus I} p_{(I,s)}\, p_{J \setminus \{s\}}\, \varepsilon(s, J \setminus \{s\})$$

$$= \sum_{\ell=1}^{k+1} (-1)^\ell p_{(i_1, \dots, i_{k-1}, j_\ell)}\, p_{(j_1, \dots, \hat{j_\ell}, \dots, j_{k+1})} \qquad (= 0 \text{ if not all distinct})$$

Example: $G(2,4)$

- $I=(1) \qquad J=(2,3,4)$

$$p_{12} \cdot p_{34} - p_{13} p_{24} + p_{14} \cdot p_{23} = 0$$

- $I=(2) \qquad J=(1,2,3)$

$$p_{21} \cdot p_{23} - \underbrace{p_{22}}_{\overset{\shortparallel}{0}} p_{13} + p_{23} \cdot p_{12} = 0$$

Remarks:

$$K[p_S,\ \#S=k] \longrightarrow K\left[(a_{ij})\ \begin{smallmatrix} 1\leq i\leq k \\ 1\leq j\leq n \end{smallmatrix}\right]$$

$$p_{(s_1,\dots,s_k)} \longmapsto \det\left(A_{(s_1,\dots,s_k)}\right)$$

k minor of matrix $A=\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{k1} & \dots & a_{kn} \end{bmatrix}$

the relations can be proved using k-minors

e.g. $\Delta_{(1,2)}\Delta_{(3,4)} - \Delta_{(1,3)}\Delta_{(2,4)} + \Delta_{(1,4)}\Delta_{(2,3)} = 0$.

Coordinate ring

The homogeneous ideal defining the image of $G(k,n)$ in $\mathbb{P}(\Lambda^k V)$ is

$$\mathcal{P} = \left\langle \alpha_{I,J}\ ;\ \begin{matrix} I\subset [\![1,n]\!],\ \#I=k-1 \\ J\subset [\![1,n]\!],\ \#J=k+1 \end{matrix} \right\rangle$$

hence $\quad G(k,n) = \mathrm{Proj}(\mathcal{B})$

where $\mathcal{B} = K[p_S\ ;\ \#S=k]/\mathcal{P}$

Another point of view:

$GL_k \curvearrowright S(k,n)$ $\qquad\qquad$ $G(k,n) = S(k,n)/GL_k$

By invariant theory, invariant polynomials of $K[(a_{ij})] = K[S(k,n)]$ are polynomials in the k-minors of $A = [a_{ij}]_{ij}$.

Get an isomorphism of graded K-algebras:

$$\begin{array}{ccc} \mathcal{B} = K[p_S]/\mathcal{P} & \longrightarrow & \text{invariant polynomials} \\ p_S & \longmapsto & \det(A_S) \end{array}$$

Graded part of degree d in $K[\cancel{p_S]}(a_{ij})]^{GL_k}$.

polynomials in $K[S(k,n)]$ st $\forall g \in GL_k$, $g \cdot P = \det(g)^d P$.

Ulysse

Goal: Parametrize subvarieties $X \subset \mathbb{P}^{n-1}$

1st case: Hypersurfaces $X \subset \mathbb{P}^{n-1}$

$$\{\text{hypersurface of degree } d\} \xleftrightarrow{1:1} \mathbb{P}(S^d \mathbb{C}^n)$$
$$V(f) \longleftarrow f$$

2nd case: Projective subspaces

$$\left\{\begin{matrix}\text{projective subspace} \\ \text{of dim } k-1\end{matrix}\right\} \longleftrightarrow G(k,n)$$
$$\mathbb{P}(M) \longleftarrow M$$

We consider irreducible closed subvarieties $X \subset \mathbb{P}^{n-1}$ of dimension $k-1$ and degree d.

Plan: Associate to X a hypersurface $Z(X) \subset G(n-k,n)$ of degree d and $Z(X) = V(R_X)$, R_X the X-resultant.

$\mathcal{B}_d$

① **Preliminaries**

a) **Degree in $\mathbb{P}^{n-1}$**

Defn: The **degree** of $X^{k-1} \subset \mathbb{P}^{n-1}$ is $\#(X \cap M)$ where $M \subset \mathbb{P}^{n-1}$ is a generic projective subspace of dimension $n-k$

Ex: $\deg(V(f)) = \# V(f|_{\mathbb{P}^1}) = \deg(f)$ for hypersurfaces

b) Degree of a hypersurface in $G(k,n)$

Consider $N \subset M \subset \mathbb{P}^{n-1}$ flag of projective subspaces of dimension $k-2$ and k.

Definition: $P_{NM} := \{ L \in G(k,n) \, ; \, N \subset L \subset M \}$ pencil associated to N, M.

$$\simeq \mathbb{P}(M/N) \simeq \mathbb{P}^1$$

Defn: The **degree** of a hypersurface $Z \subset G(k,n)$ is $\#(Z \cap P_{NM})$ for $N \subset M$ generic.

c) Irreducibility

Lemma: $f : Y \longrightarrow X$ smooth (or flat) morphism of varieties. If X is irreducible and the generic fiber F is irreducible, then Y is irreducible, and

$$\dim Y = \dim X + \dim F$$

② Hypersurfaces in $G(k,n)$

Prop: Let $Z \subset G(k,n)$ be a hypersurface, irreducible of degree d. Then, $\exists f \in \mathcal{B}_d$, unique up to a constant factor, such that $Z = V(f)$.

Proof: Recall

$$\begin{tikzcd} S(k,n) \arrow[d, "p"] & \& & K[a_{ij}] \\ G(k,n) & & & \mathcal{B} \arrow[u] \end{tikzcd}$$

Define $\tilde{Z} = \overline{p^{-1}(Z)} \subset M_{k,n}$, and $f \in K[a_{ij}]$ the equation of $\tilde{Z}$.

($\tilde{Z}$ is irreducible hypersurface by the lemma).

$\forall g \in GL_k \,,\ g^{\pm 1}\tilde{Z} = \tilde{Z} \Rightarrow \exists \chi(g) \in K^\times$ st $f(gA) = \chi(g) f(A)$.

It defines a character $\chi : GL(k) \to \mathbb{G}_m$

$\Rightarrow \exists\, d' \in \mathbb{Z}$ st $\chi(g) = \det(g)^{d'}$ (in fact $d' > 0$)

$\Rightarrow f \in \mathcal{B}_{d'}$.

Claim: $d' = d$

$\to$ Prove that $\#(P_{NM} \cap V(f)) = d'$ for $f \in \mathcal{B}_{d'}$

$$\begin{gathered} \| \\ \# V(f|_{P_{NM}}) \end{gathered}$$

$f|_{P_{NM}}$ is a section of $\mathcal{O}_{G(k,n)}(d')\big|_{P_{NM}}$

have to show (later)