GT GKZ – Sylvain (15/04/26)

1 The discriminant as the determinant of a complex

$X \subseteq \mathbb{P}(V^*)$, $X^\vee \subseteq \mathbb{P}(V)$ the dual variety

Δ_X the discriminant s.t. $X^\vee = \{\Delta_X = 0\}$ if $X^\vee$ is a hypersurface

$\Delta_X = 1$ otherwise

(only well-defined up to a scalar)

$\mathcal{L} = \mathcal{O}_X(1)$, $J(\mathcal{L})$ the jet bundle. We have a map $V \to H^0(X, \mathcal{L}) \to H^0(X, J(\mathcal{L}))$
$f \mapsto f \mapsto j(f)$

Prop: $f \in V \setminus \{0\}$
$[f] \in X^\vee \iff \exists x \in X,\ j(f)_x = 0$

To $j(f)$ we can associate Koszul complexes
$\mathcal{K}_+(J(\mathcal{L}), j(f)) : \dots$
$\mathcal{K}_-(J(\mathcal{L}), j(f)) : \dots$

Prop: $[f] \in X^\vee \iff \mathcal{K}_+$ is not exact $\iff \mathcal{K}_-$ is not exact.

We can choose a line bundle $\mathcal{M}$ such that $\mathcal{K}_\pm$ exact $\Rightarrow \Gamma(\mathcal{K}_\pm \otimes \mathcal{M})$ is exact.
Let $C_\pm(X, \mathcal{M}) = \Gamma(\mathcal{K}_\pm(J(\mathcal{L}), j(f)) \otimes \mathcal{M})$ (or $C_\pm(f)$)

If $[f] \notin X^\vee$ then $C_\pm(f)$ is exact. Fix $\underline{e}$ a basis of $C_\pm$
$\rightsquigarrow \det(C_\pm(f)) \in k^*$.
$\qquad \| $
$\quad \Delta_\pm(f)$

Let $S = S^\cdot(V^*)$ (if $V = k^{n+1}$ then $S = k[x_0, x_1, \dots, x_n]$).
Then $\Delta_\pm \in \mathrm{Frac}(S)$
$\qquad \qquad \qquad \qquad \ni f$

Instead of considering the map $f \mapsto \Delta_\pm(f)$, we look at the universal complex
$C_\pm \otimes_k S$ (complex of free S-modules of finite rank).

This complex is generically exact, i.e. $(C_\pm \otimes_k S) \otimes_S K$ is exact ($K = \mathrm{Frac}(S)$).
[Running assumption: $X \neq$ point $\leadsto X^\vee \neq \mathbb{P}(V)$.]
But $C_\pm \otimes_k S$ is not exact in general.

$$\Delta_\pm = \det\left((C_\pm \otimes_k S) \otimes_S K, \partial, \underline{e}\right) \in K^* \quad \text{(homogeneous)}$$

<u>Remark</u>: If $\Delta_\pm = \frac{P}{Q} \in K^*$, we have seen that

$$[f] \notin X^\vee \Longrightarrow P(f)Q(f) \neq 0$$

If $X^\vee$ is a hypersurface, we therefore have

$$V(PQ) \subseteq X^\vee = V(\Delta_X).$$

$$\Rightarrow \Delta_X \in \sqrt{(PQ)} \underset{\Delta_X \text{ irred}}{\Longrightarrow} \Delta_\pm = \Delta_X^{\alpha_\pm} \text{ for some } \alpha_\pm \in \mathbb{Z}.$$

<u>Main theorem</u>: If X is smooth, $\Delta_- = \dfrac{1}{\Delta_X}$, $\Delta_+ = \Delta_X^{(-1)^{\dim X}}$

2 Reformulation of the problem

By the last talk, $\forall \pi \in S$ irreducible, $\mathrm{ord}_\pi(\Delta_\pm) = \sum_i (-1)^{i+1} \lg_{S_{(\pi)}}\left(H^i(C_\pm \otimes_k S)_{(\pi)}\right)$
It is enought to prove:

<u>Prop</u>:
(a) $\forall i$ except at the extreme right of $C_\pm$, $\forall \pi \in S$ irreducible, $H^i(C_\pm \otimes_k S)_{(\pi)} = 0$
(b) For the last i, i.e. $i=0$ for C_-, $i = n = \dim X + 1$ for C_+, $\forall \pi \in S$,

$$\begin{cases} H^i(C_\pm \otimes_k S) = 0 & \text{if } \pi \neq \Delta_X \\ \lg H^i(C_\pm \otimes_k S) = 1 & \text{if } \pi = \Delta_X \end{cases}$$

To prove this, we consider complexes of sheaves on $\mathbb{P}(V)$ associated to $C_\pm \otimes S$.

Each $C_+^j \otimes_k S$ is a graded module

$\rightsquigarrow \widetilde{C_+^j \otimes_k S}$ sheaf on $\operatorname{Proj} S = \mathbb{P}(V)$

$\partial_f : C_+^j \otimes_k S \longrightarrow C_+^{j+1} \otimes_k S$ has degree 1

$$\Longrightarrow \quad \widetilde{C_+^j \otimes_k S} \longrightarrow \widetilde{C_+^{j+1} \otimes_k S} \otimes \mathcal{O}(1)$$

This defines a complex of sheaves on $\mathbb{P}(V)$,

$$\mathcal{O}(C_+) : 0 \to \underbrace{\widetilde{C_+^0 \otimes_k S}}_{\mathcal{O}_{\mathbb{P}(V)}} \longrightarrow \widetilde{C_+^1 \otimes_k S} \otimes \mathcal{O}(1) \longrightarrow \cdots \to \widetilde{C_+^n \otimes_k S} \otimes \mathcal{O}(n) \to 0$$

Similarly, we have $\mathcal{O}(C_-)$.

By exactness of the functor "fiber at x", the fiber at $x \in \mathbb{P}(V)$ of the coherent sheaves are

$$\underline{H}^i(\mathcal{O}(C_\pm))_x \simeq H^i(C_\pm \otimes_k S)_x$$

3) Reminders on the incidence variety

$$W_X = \{(x,H) \in X \times \mathbb{P}(V) \mid x \in X,\ H \text{ is a hyperplane tangent to } X \text{ at } x\}$$

$$\begin{tikzcd}
& W_{X^\vee} = W_X \arrow[d, hook, "i"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_2"] \arrow[drr, "q"] & & \\
& X \times \mathbb{P}(V) \arrow[dl, "p_X"'] \arrow[dr, "p_V"] & & X^\vee \arrow[dl, hook, "j"] \\
X & & \mathbb{P}(V) &
\end{tikzcd}$$

Properties:

1) If X is smooth then W_X is smooth and $\mathrm{codim}_{X\times\mathbb{P}(V)}(W_X) = \dim X + 1$.

2) If $X^\vee$ is a hypersurface then q is birational.

Sketch of proof:

1) $\forall x \in X$, $p_1^{-1}(x)$ is a $\mathbb{P}^\ell$ where $\ell = \mathrm{codim}(X) - 1$.

$\Rightarrow$ W_X is fibered in projective spaces

$\Rightarrow$ p_1 is smooth

$\Rightarrow$ W_X is smooth

And $\dim W_X = \dim X + \ell$.

2) Since $W_X = W_{X^\vee}$, q is an isomorphism over $(X^\vee)_{reg}$.

4 A universal Koszul complex

We have a map

$$\begin{array}{ccccc} V & \longrightarrow & H^0(X,\mathcal{L}) & \longrightarrow & H^0(X,J(\mathcal{L})) \\ f & \longmapsto & f & \longmapsto & j(f) \end{array}$$

It induces

$$\begin{array}{ccc} V\otimes V^* & \longrightarrow & H^0(X\times\mathbb{P}(V),\ p_X^*J(\mathcal{L})\otimes p_V^*\mathcal{O}_{\mathbb{P}(V)}(1)) \\ f\otimes\varphi & \longmapsto & p_X^*j(f)\otimes p_V^*\varphi \end{array}$$

Let $\sigma \in H^0(X\times\mathbb{P}(V),\ p_X^*J(\mathcal{L})\otimes p_V^*\mathcal{O}_{\mathbb{P}(V)}(1))$ denote the image of $id \in V\otimes V^*$.

Remark: σ is universal in the following sense:

$\forall f \in V\setminus\{0\}$, we have $[f_X]^*\sigma = j(f)$.

$$\begin{tikzcd} X \arrow[r, "{[f]_X}"] \arrow[d] & X\times\mathbb{P}(V) \arrow[d] \\ \mathrm{Spec}\,k \arrow[r, "{[f]}"] & \mathbb{P}(V) \end{tikzcd}$$

Let $E = p_X^* J(\mathcal{L}) \otimes p_V^* \mathcal{O}_{\mathbb{P}(V)}(1)$. We have the Koszul complexes $\mathcal{K}_\pm(E,\sigma)$.

<u>Remark</u>: $Z(\sigma) = W_X$.

$$\mathcal{K}_+(E,\sigma): \quad 0 \to \underset{(0)}{\mathcal{O}_{X\times\mathbb{P}(V)}} \to \underset{(1)}{p_X^* J(\mathcal{L}) \otimes p_V^* \mathcal{O}_{\mathbb{P}(V)}(1)} \to \cdots \to \underset{(r)}{p_X^* \Lambda^{r} J(\mathcal{L}) \otimes p_V^* \mathcal{O}(r)} \to 0$$

By Matthieu's talk, $\mathcal{K}_\pm(E,\sigma)$ are exact except at the right, and

$$\begin{cases} \underline{H}^0(\mathcal{K}_-) = i_*(\mathcal{O}_{W_X}). \\ \underline{H}^r(\mathcal{K}_+) = i_* i^*(\Lambda^r E) \end{cases}$$

<u>Key point / proposition</u>: $$\begin{cases} Rp_{2*}(p_1^*\mathcal{M}) \simeq \mathcal{O}(C_-) \\ Rp_{2*}(p_1^*\mathcal{M} \otimes \mathcal{E})[-r] \simeq \mathcal{O}(C_+) \end{cases}$$
where $\mathcal{E} = i_*(p_X^* \Lambda^r J(\mathcal{L}) \otimes p_V^* \mathcal{O}(r))$

Proof:
[Skipped; essentially the projection formula + homological algebra.]

End of the proof of the main theorem: We need to prove that the fibers of the homology sheaves of $\mathcal{O}(C_\pm)$ at $\pi \in S$ are of length 1 at the extreme right if $\pi = \Delta_X$, and are zero otherwise.

This follows from this: let $\cancel{\mathcal{L}}$ $\mathcal{N}$ be a line bundle on W_X.

$$Rp_{2*}\mathcal{N} = R(j\circ q)_* \mathcal{N} = Rj_* Rq_* \mathcal{N} = j_* Rq_* \mathcal{N}$$

If $\pi \neq \Delta_X$ then the point $p = (\pi)$ of $\mathbb{P}(V)$ (the generic point of $Z(\pi)$) is not in $X^\vee$.

$$\Rightarrow (R^i p_{2*}\mathcal{N})_p = 0 \quad \forall i$$

If $\pi = \Delta_X$ then $p = (\pi)$ is the generic point of $X^\vee$, and

$$(R^i p_{2*} \mathcal{N})_p = (j_* R^i q_* \mathcal{N})_p = (R^i q_* \mathcal{N})_p = \mathcal{O}_{X^\vee, p} \quad \text{length } 1.$$

$(i=0)$

q birational