Damien GTGKZ 01/04/2026

Summary

$X \subset \mathbb{P}(V^*)$ $\quad \mathcal{L} = \mathcal{O}_X(1)$

$f \in V$ viewed as a section of $\mathcal{L}$

$j(f) \in J(\mathcal{L})$

$K_{\pm} := K_{\pm}(J(\mathcal{L}), j(f))$

$K_+ = (\Lambda^{\bullet}_{\mathcal{O}_X} J(\mathcal{L}), \, - \wedge j(f))$

$K_- = (\Lambda^{\bullet}_{\mathcal{O}_X} J(\mathcal{L})^*, \, \iota_{j(f)})$ with differential ∂_f

Thm: $[f] \in X^\vee \iff K_+$ is not exact

$\iff K_-$ is not exact

Remark: if $K^\bullet$ is a complex of sheaves (of $\mathcal{O}_X$-modules) and $\mathcal{M}$ is a line bundle, then

$$K^\bullet \text{ is exact} \iff K^\bullet \otimes_{\mathcal{O}_X} \mathcal{M} \text{ is exact}$$

ETS $\Rightarrow$

+ exactness is a local property & $\mathcal{L} \underset{loc}{\simeq} \mathcal{O}_X$

Choose $\mathcal{M}$ ample. Then $\forall F$ coherent sheaf, there exist $n \in \mathbb{N}$ st $\forall i > 0$, $H^i(X, F \otimes \mathcal{M}^{\otimes n}) = 0$.

Then $\exists n \gg 0$ st $\forall i > 0, \forall j$, $H^i(X, K^j \otimes \underbrace{\mathcal{M}^{\otimes n}}_{N})=0$

provided that $K^\bullet$ is bounded

<u>Lemma</u>: Let $\mathcal{F}^\bullet$ a complex of sheaves st:

a) $\mathcal{F}^\bullet$ is exact

b) $\forall i > 0$, $\forall j$, $H^i(X, \mathcal{F}^j) = 0$

Then $\Gamma(X, \mathcal{F}^\bullet)$ is exact.

complex of global section

<u>Idea of proof</u>: Let $\mathcal{U} = (U_\alpha)_\alpha$ be an open cover of X by affines.

Consider the complexes $\check{C}(\mathcal{U}, \mathcal{F}^j)$

i.e. $\check{C}^k(\mathcal{U}, \mathcal{F}^j) = \prod_{\alpha_0 \dots \alpha_k} \Gamma(U_{\alpha_0 \dots \alpha_k}, \mathcal{F}^j)$

These complexes $\check{C}^k(\mathcal{U}, \mathcal{F}^\bullet)$ are exact for every k. ($\mathcal{F}^\bullet$ exact + working on affines)

Consider the total complex of the double complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$

1^{st} spectral sequence:

$$E_1^{\bullet,\bullet} = E_1^{0,\bullet} = \Gamma(X, \mathcal{F}^\bullet)$$ (using the assumption b)

$$E_2^{\bullet,\bullet} = E_\infty^{\bullet,\bullet} = E_\infty^{0,\bullet} = H^\bullet(\Gamma(X,\mathcal{F}^\bullet))$$

2^{nd} spectral sequence:

$$E_1^{\bullet,\bullet} = 0$$

$\square$

Consider $\Gamma(X, \mathcal{F}_\pm)$ where $\mathcal{F}_\pm = K_\pm \otimes_{\mathcal{O}_X} \mathcal{N}$ $\quad \mathcal{N} = \mathcal{M}^{\otimes n}$ $n \gg 0$.

Notation: $\Gamma(X, \mathcal{F}_\pm) = (C_\pm^\bullet(X, \mathcal{N}), \partial_f)$

Pick a basis of $C_\pm^\bullet(X, \mathcal{N})$, $e_\pm$ (not related with f) $\quad \mathrm{Frac}(V)/\mathbb{K}^* \ni \psi$

There exists a non-zero scalar $\det(C_\pm^\bullet(X,\mathcal{N}), \partial_f, e_\pm) =: \Delta^\pm_{X,\mathcal{N}}(f)$

→ this is a rational function on V (as a function of f)

→ change of base $e_\pm$ multiplies this by a nonzero constant.

Caveat: $\det(\ldots)$ is associated with an exact complex with a basis.
So for $\Delta^\pm_{X,\mathcal{N}}(f)$ to be defined, one needs to assume that $[f] \notin X^\vee$

Thm: Under the previous assumptions,

$$\Delta^-_{X,\mathcal{N}}(f) = \Delta_X(f) \quad \text{and} \quad \Delta^-_{X,\mathcal{N}}(f)^{(-1)^{\dim(X)+1}} = \Delta_X(f)$$

Goal: define det (...)

Cone of a morphism of complexes

$f : V \longrightarrow W$ morphism of complexes

$C = \mathrm{Cone}(f)$ is defined as:

$$C^i = W^i \oplus V^{i+1}$$

$$d_C(w,v) = \left(d_W(w) + (-1)^{i+1} f(v),\ d_V(v) \right)$$

(think of $\begin{bmatrix} d_W & (-1)^{i+1} f \\ 0 & d_V \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}$

If $g : W \longrightarrow Z$ is another morphism of complexes
and h^i a null homotopy for $g \circ f$: $(h^i : V^i \to Z^i)_{i \in \mathbb{Z}}$ st

$$g \circ f(v) = h d_V(v) + d_Z h(v)$$

Then there is a factorization
$$\begin{tikzcd} V \arrow[r, "f"] & W \arrow[r, hook] \arrow[dr, "g"'] & \mathrm{Cone}(f) \arrow[d, "\tilde{g}"] \\ & & Z \end{tikzcd}$$

$$\tilde{g}(w,v) = g(w) + (-1)^{|v|+1} h(v)$$

There is an exact sequence of complexes

$$0 \longrightarrow W \longrightarrow \mathrm{Cone}(f) \longrightarrow V[1] \longrightarrow 0$$

Determinants

- W vector space of dim $n \in \mathbb{N}$, $\mathrm{Det}(W) \overset{\mathrm{def}}{=} \Lambda^n W$
- W finite dimensional graded vector space, $\mathrm{Det}(W) = \bigotimes_{n \in \mathbb{Z}} \mathrm{Det}(W^n)^{(-1)^n}$

Obvious properties

* Det is a functor $(\mathrm{Vect}^{\mathrm{gr, f.d.}}, \mathrm{iso}) \longrightarrow (\mathrm{Lines}, \mathrm{iso})$
* multiplication by $\lambda \in \mathbb{K}^*$ is sent to multiplication by $\lambda^{\chi(W)}$

  where $\chi(W) = \sum_{n \in \mathbb{Z}} (-1)^n \dim(W^n)$
* $\mathrm{Det}(V \oplus W) = \mathrm{Det}(V) \otimes \mathrm{Det}(W)$
* $\mathrm{Det}(W[1]) = \mathrm{Det}(W)^*$

Euler isomorphism

Assume that $W^\bullet$ is a finite dim complex. There is a canonical isomorphism

$$\underbrace{\mathrm{Det}(W^\bullet)}_{\text{seen as graded vector space}} \xrightarrow{\mathrm{Eu}} \mathrm{Det}(H^\bullet(W^\bullet))$$

Proof of this claim

Lemma: if $0 \to A \to B \to C \to 0$ is an exact sequence then there is a canonical isomorphism $\mathrm{Det}(B) \simeq \mathrm{Det}(A) \otimes \mathrm{Det}(C)$

<u>Proof</u>: Choose a splitting $C \hookrightarrow B$, then we get

$$\mathrm{Det}(A) \otimes \mathrm{Det}(C) \xrightarrow{\sim} \mathrm{Det}(B)$$

and this does not depend on the choice of splitting. $\square$

There are two exact sequences:

$$0 \longrightarrow \mathrm{Ker}(d_i) \longrightarrow W^i \longrightarrow \mathrm{Im}(d_i) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}(d_{i-1}) \longrightarrow \mathrm{Ker}(d_i) \longrightarrow H^i(W) \longrightarrow 0$$

By the lemma,

$$\mathrm{Det}(W) \simeq \mathrm{Det}(\mathrm{Ker}(d)) \otimes \mathrm{Det}(\mathrm{Im}(d))$$

due to the shift

$$\simeq \left(\mathrm{Det}(H^\bullet(W)) \otimes \mathrm{Det}(\mathrm{Im}(d))^{*}\right) \otimes \mathrm{Det}(\mathrm{Im}(d))$$

<u>Corollary</u>: If W is exact, then $\mathrm{Det}(W) \simeq \mathbb{K}$ canonically

<u>Corollary</u>: $\mathrm{Det}(\mathrm{Cone}(f)) \simeq \mathrm{Det}(W) \otimes \mathrm{Det}(V)^*$

From now on, W will be assumed to be exact.

Let $\underline{e}$ be a basis of W.

There is an element $\mathrm{Det}(e) \in \mathrm{Det}(W) \xrightarrow{\mathrm{Eu}} \mathrm{Det}(H^\bullet(W)) \underset{\text{canon}}{\simeq} \mathbb{K}$

$$\mathrm{Det}(e) \longmapsto \boxed{\det(W, d_W, e) \in \mathbb{K}^*}$$

Examples :

- 2 term complex $W^{-1} \xrightarrow{d} W^0$ exact $\Leftrightarrow$ d is an isomorphism

$\underline{e}$ basis : $\underline{e_{-1}}$ basis of W^{-1} and $\underline{e_0}$ basis of W^0

$$\det(W^\bullet, d, \underline{e}) = \det\left(M_{\underline{e_{-1}}, \underline{e_0}}(d)\right)$$

det has nice properties :

* sends $(\oplus, \sqcup)$ to products.

* if $\underline{e(i)}$ and $\underline{f(i)}$ are basis of W^i and $A(i)$ transition matrix

i.e $f(i)_p = \sum_q A(i)_{p,q} e(i)_q$

$$\det(W, d, \underline{f}) = \det(W, d, \underline{e}) \prod_{i \in \mathbb{Z}} \det(A(i))^{(-1)^i}$$

Another example :

- 3 term complex $\left(0 \to A \xrightarrow{d_{-1}} B \xrightarrow{d_0} C \to 0\right)^{=W^\bullet}$ exact

degrees -1 0 1

$\underline{e} = (\underline{a}, \underline{b}, \underline{c})$ basis of A, B, C

D_{-1} matrix of d_{-1} in bases $\underline{a}$ and $\underline{b}$, of size $n \times m$ & rank m

D_0 —— d_0 —— $\underline{b}$ — $\underline{c}$

There is an invertible $m \times m$ submatrix of D_{-1}, denoted by $\overline{D}_{-1}$

$p := n-m$, there is an invertible $p \times p$ submatrix of D_0, denoted by $\overline{D}_0$.

$$0 \to A \to B \to C \to 0$$

$$\wr\wr$$

$$0 \to A \xrightarrow[\overline{D}_{-1}]{} \mathrm{Im}(d_{-1}) \to 0 \quad \oplus \quad 0 \to \overline{B} \xrightarrow[\overline{D}_0]{} C \to 0$$

We get that: $\det(W, d, \underline{e}) = \dfrac{\det(\bar{D}_{-1})}{\det(\bar{D}_0)}$

Cayley formula (next week)