$\boxed{\text{GT GKZ}}$ 8 avril Damien

$W^\bullet$ graded vector space $\rightsquigarrow$ $\det(W^\bullet)$ line

$\underline{e}$ basis of $W^\bullet$ $\rightsquigarrow$ $\det(\underline{e}) \in \det(W^\bullet)$

$(W^\bullet, d)$ exact complex $\quad \det(\underline{e}) \in \det(W^\bullet) \xrightarrow{\sim} \mathrm{Det}(H^\bullet(W^\bullet, d)) = \mathbb{K}$

$$\det(\underline{e}) \longmapsto \det(W, d, \underline{e}) \in \mathbb{K}$$

<u>Example 2 term complex</u> $\quad W^{-1} \xrightarrow[d]{} W^0 \quad$ exact $\Leftrightarrow$ d is an isom

$$\det(W, d, \underline{e}) = \det\left(M_{\underline{e}_{-1}, e_0}(d)\right)^{-1}$$

correction wrt last time & the book

$$\det(W^\bullet) = \det(W^{-1})^* \otimes \det(W^0) \xrightarrow[(d^*)^{-1} \otimes \mathrm{id}]{} \det(W^0)^* \otimes \det(W^0) \simeq \mathbb{K}$$

$\mathrm{Det}(\underline{e}) = \mathrm{Det}(\underline{e}_{-1}^*) \otimes \det(\underline{e}_0) \qquad \det(M_{\underline{e}_{-1}, e_0}(d))^{-1} \mathrm{Det}(\underline{e}_0^*) \otimes \mathrm{Det}(\underline{e}_0)$

in general: $V^\bullet \xrightarrow[\text{isom}]{f} W^\bullet \qquad \underline{e}$ basis of V, $\underline{e}'$ basis of W

$$\begin{tikzcd} & \mathrm{Det}(V^\bullet) \arrow[d, "\mathrm{Det}(f)", "\sim"'] \\ \mathbb{K} \arrow[ur, "\mathrm{Det}(\underline{e})"] \arrow[dr, "\mathrm{Det}(\underline{e}')"'] & \\ & \mathrm{Det}(W^\bullet) \end{tikzcd}$$

$$\mathrm{Det}(f)\,\mathrm{Det}(\underline{e}) = \det M_{e,e'}(f)\,\mathrm{Det}(e')$$

<u>Example 3 term complex</u> $\quad W^\bullet = \quad \cdots \to 0 \to \overset{-1}{A} \xrightarrow{d_{-1}} \overset{0}{B} \xrightarrow{d_0} \overset{1}{C} \to 0 \to \cdots$

$\underline{a} \quad \underline{b} \quad \underline{c}$

$D_{-1} = M_{\underline{a}, \underline{b}}(d_{-1}) \qquad D_0 = M_{\underline{b}, \underline{c}}(d_0)$

$\bar{D}_{-1} \qquad \bar{D}_0 \qquad$ square matrices, invertible as last time

$$\det(W^\bullet, d, \underline{a}\,\underline{b}\,\underline{c}) = \frac{\det(\bar{D}_0)}{\det(\bar{D}_{-1})}$$

correction wrt last time

$$W^\bullet = \to 0 \to k^{B_0} \xrightarrow[D_0]{} k^{B_1} \xrightarrow[D_1]{} \cdots \xrightarrow[D_{n-1}]{} k^{B_n} \to 0 \to \cdots \quad \text{exact complex}$$

each $\underbrace{B_i}_{}$ is a totally ordered finite set

index set for a basis of $W^i = k^{B_i}$

Defn: A collection $(I_i)_{i=0,\dots,n}$ of subsets $I_i \subset B_i$ is admissible if:

$I_0 = \emptyset$ $\qquad \forall i < n, \ (D_i)_{B_i \setminus I_i, I_{i+1}}$ is invertible

$I_n = B_n$

$(\Rightarrow \#(B_i \setminus I_i) = \# I_{i+1})$

Proposition: Admissible collections exist.

Proof: by induction. D_0 is injective

can remove $B_1 \setminus I_1$ vectors in D_0 so that $(D_0)_{B_0, I_1}$ is invertible

$D_1|_{k^{B_1 \setminus I_1}}$ is injective (by exactness) $\to$ repeat the process.

$\square$

Pick an admissible collection and define

$$\Delta_i = \det\left((D_i)|_{B_i \setminus I_i, I_{i+1}}\right) \in K^*$$

$$\boxed{\textbf{Theorem}: \det(W^\bullet, D, B_\cdot) = \prod_{i=0}^{n-1} \Delta_i^{(-1)^i}}$$

Sketch of proof: $W^\bullet \simeq k^{B_0} \xrightarrow{\sim} k^{I_1}$

$\oplus$

$k^{B_1 \setminus I_1} \longrightarrow k^{I_2}$

$\oplus$

$\cdots$

more precisely for based complexes:

$$0 \to U \to W \to V \to 0$$

with $V = k^{B_0} \longrightarrow k^{I_1}$

$U = 0 \to k^{B_1 - I_1} \longrightarrow k^{B_2} \to \cdots \to k^{B_n} \to 0$

$$\det(W^\cdot, B) = \det(U_\cdot, (B_1 - I_1, B_2, \dots, B_n)) \times \underbrace{\det(U, (B_0, I_1))}_{\Delta_0}.$$

⚠ there is actually a sign $\varepsilon(I_\cdot)$ in the formula

e.g. $0 \to k \xrightarrow[\binom{1}{1}]{} k^2 \xrightarrow[(1\ -1)]{} k \to 0$ depending on the choice of I_1, get a different sign ± 1.

□

Filtrations

Assume we have a finite filtration $\cdots \subset F_n W^\cdot \subset F_{n+1} W^\cdot \subset \cdots$

Proposition: $\mathrm{Det}(W) = \bigotimes_{j \in \mathbb{Z}} \left(gr_j^F(W^\cdot) \right)$ where $gr_j^F(W) = F_j W / F_{j-1} W$

Assume now that $\underline{e}$ is a basis adapted to the filtration, and that d preserves the filtration.

Denote by $gr(\underline{e})$ the induced basis of $gr_\cdot^F(W)$.

Proposition: $(W^\cdot, F, d)$ is a finitely filtered complex of finite dimension, as above.

① if $gr_j^F(W)$ is exact $\forall$ for every j, then $W^\cdot$ is exact

② $\det(W^\cdot, d, \underline{e}) = \prod_{j \in \mathbb{Z}} \det(gr_j^F(W), gr^F(d), gr(\underline{e}))$

Base change formula

If $R = \mathcal{O}(X)$ ring of functions on an affine alg. var X

k a R-algebra

(M,d) complex of free R-modules

$\underline{e}$ basis

Then
$$\boxed{\begin{array}{ccc} \det(M,d,\underline{e}) & = & \det\left(M \otimes_R k, \, d \otimes 1, \underline{e}\right) \\ \Uparrow & & \Uparrow \\ R^* & & k^* \end{array}}$$

Sylvain

Length of a module

Defn: A a ring, $M \in (A\text{-Mod})$. M has finite length if
$\exists\ M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$. "composition series"
st M_i / M_{i+1} is simple (non nontrivial submodule)

Prop: If M has finite length, all composition series have the same length n, denoted by $\lg_A(M)$.

Rem 1: finite length $\Rightarrow$ finite type

2: $\lg_A(\cdot)$ is additive on short exact sequences:

$$0 \to M_1 \to M_2 \to M_3 \to 0 \quad \Rightarrow \quad \lg_A(M_2) = \lg_A(M_1) + \lg_A(M_3).$$

Ex 1: $A = k$ $\quad \lg_A = \dim_k$

2: A local noetherian ring, $M \in (A\text{-Mod})^{f.t.}$ has finite length iff
$\exists i \in \mathbb{N},\ m^i M = 0$ where m maximal ideal of A
and $\lg_A(M) = \sum_{i \geq 0} \dim_k \left(m^i M / m^{i+1} M \right)$

3: $A = \mathbb{Z}$ $\quad M = \prod_i \mathbb{Z}/d_i\mathbb{Z}$ has finite length if $\forall i,\ d_i \neq 0$
finite product
and then $\lg_{\mathbb{Z}}(M) = \sum_i \sum_{p \in \mathcal{P}} v_p(d_i)$

Let R be a noetherian integral domain, regular and factorial. $k = \mathrm{Frac}(R)$

If $\pi \in R$ is irreducible, $\mathrm{ord}_\pi(a) = v_\pi(a)$ for $a \in k^* :=$

the unique integer st $a = \varepsilon \prod_{\pi \in \mathcal{P}} \pi^{v_\pi(a)}$, $\varepsilon \in R^\times$

Let $M^\cdot$ be a finite complex of finite free R-modules.

Assume that M is <u>generically exact</u>, that is $M \otimes_R k$ is exact.

Let e be an R-basis of M.

$\rightsquigarrow \det(M \otimes_R k, d, e) \in k^*$

<u>Rem</u>: If M is exact, it belongs to $R^\times$ by the base change formula.

We want to relate $\det(\cdot)$ and the $H^i(M^\cdot)$

<u>Example</u>: $R = \mathbb{Z}$

$$M = 0 \to \underset{\text{degree } 0}{\mathbb{Z}^2} \xrightarrow[\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}]{} \mathbb{Z}^2 \to 0 \qquad \text{canonical basis}$$

$\det(M^\cdot) = \det(D) = 12$

$H^0(M) = 0 \qquad H^1(M) = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

$\# H^1(M) = |\det(M)|$

better: $\forall p \in \mathcal{P}$, $\mathrm{ord}_p(\det M) = \lg_{\mathbb{Z}_{(p)}} \big(\underbrace{H^1(M) \otimes_{\mathbb{Z}} \overset{\text{localization}}{\mathbb{Z}_{(p)}}}_{= \begin{cases} \mathbb{Z}/(3) & \text{if } p = 3 \\ \mathbb{Z}/(4) & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}} \big) = \begin{cases} 1 & \text{if } p = 3 \\ 2 & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}$

More generally, $\forall \quad \overset{d^\circ 0}{\mathbb{Z}^n} \xrightarrow{D} \overset{1}{\mathbb{Z}^n}$ st $\det D \neq 0$

then $v_p(\det(M)) = \lg\left(H^1(M) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}\right)$

Theorem: With the above assumptions, M generically exact, then

$\forall \pi \in R$ irreducible, $\operatorname{ord}_\pi(\det(M \otimes_R k)) = \sum_{i \in \mathbb{Z}} (-1)^{i+1} \lg_{R_{(\pi)}} \left(\underbrace{H^i(M) \otimes_R R_{(\pi)}}_{= H^i(M \otimes_R R_{(\pi)})}\right)$

Rem: If e and e' are two R-basis of M, then

$$\det(M \otimes_R k, e) = \varepsilon \det(M \otimes_R k, e') \quad \text{for some } \varepsilon \in R^\times.$$

so $\operatorname{ord}_\pi(\det(M \otimes_R k))$ does not depend on the choice of basis e.

Sketch of proof:

- Let $D^b(R)$ be the derived category of finite complexes of finitely generated R-modules.
- If $\mathcal{A}$ is a triangulated category, $K_0(\mathcal{A}) := \bigoplus_{A \in Ob(\mathcal{A})} \mathbb{Z}[A] \Big/ \substack{[B]=[A]+[C] \\ \forall A \to B \to C \to A[1] \\ \text{distinguished triangle}}$
- We may assume that $R = R_{(\pi)}$ is a dvr with uniformizer π.
- For any generically exact complex $M \in D^b(R)$ of free R-modules, we have

$$\varphi(M) = \operatorname{ord}_\pi(\det(M \otimes_R k))$$

$$\psi(M) = \sum_i (-1)^{i+1} \lg_R(H^i(M))$$

Since φ and Ψ are additive, we can extend them to $\mathcal{D}^1 \subset \mathcal{D}^b(R)$ the subcategory of generically exact complexes (even non projective modules)

Idea: take a projective resolution of M $\quad P^\bullet \xrightarrow{\sim} M^\bullet$ and set
$$\varphi(M^\bullet) = \varphi(P^\bullet).$$

Let $K_0(\mathcal{D}^1) \subset K_0(\mathcal{D}^b(R))$ be the subgroup generated by the $[M]$ for $M \in \mathcal{D}^1$.

Proposition: $K_0(\mathcal{D}^1)$ is generated by $[R/\pi R]$

Proof: Let $M \in \mathcal{D}^1$. $[M] = \sum_i \underbrace{[H^i(M)]}_{\text{torsion modules over } R}(-1)^i$

$$= \sum_i (-1)^i \sum_{j \geq 0} \underbrace{\left[\pi^j H^i(M) / \pi^{j+1} H^i(M) \right]}_{\substack{\text{f.d. } R/\pi \text{ vector space} \\ \Rightarrow \text{multiple of } [R/\pi]}} \qquad \square$$

Then to finish the proof of the Theorem, Enough to prove that
$$\varphi(R/\pi R) = \underbrace{\Psi(R/\pi R)}_{\overset{\text{direct}}{=} -1}$$

$$\varphi(R/\pi R) = \varphi\big(\big[\underset{-1}{R} \xrightarrow{\pi} \underset{0}{R}\big]\big) = \mathrm{ord}_\pi(\det[\pi])^{-1} = -1 \qquad \square$$