# Effective K-stability of spherical varieties

## Talk 2: K-stability criterion and close to Fano varieties

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# Motivation: YTD conjecture(s)

Existence of a canonical Kähler metric in $c_1(L)$ should be equivalent to an algebro-geometric notion of stability for $(X, L)$

Major examples of such results or partial results:

1. Kähler-Einstein metrics on Fano manifolds [Chen-Donaldson-Sun, Tian, 2015]
2. coupled $g$-solitons [Li-Han]
3. existence cscK (extremal) $\Rightarrow$ uniform K-stability [Berman-Darvas-Lu 2020]
4. From the analytical point of view, [Chen-Cheng 2021] proved that coercivity (modulo automorphisms) of the Mabuchi functional implies existence of cscK metrics
5. [Li 2022] some algebraic notion close to uniform K-stability $\Rightarrow$ existence cscK

**What is the "best" notion of K-stability applicable?**
(especially in direction K-stability $\Rightarrow$ existence)

# Explicit K-stability

Assume $(X, K_X^{-1})$ Fano.

1. [Li-Xu 2011] To test K-stability, enough to consider special test configurations
2. [Li, Fujita] Valuative criterion
3. [Abban-Zhuang] approach to get explicit K-stability
4. [Many people $\supset$ Cheltsov, Fujita, Shramov, Viswanathan,...] Fano threefolds

**Conjecture**

Assume now that $X$ is "close to Fano".
$\exists$ (geometrically meaningful) neighborhood $\mathcal{N}$ of $c_1(X)$ st $\forall L \in \mathcal{N}$,
$(X, L)$ is uniform K-stable iff it is K-stable wrt special test configurations.

**Goal today:** partial results in this direction for spherical varieties.

**[Donaldson 2009]** for toric surfaces, arbitrary $L$, special t.c. are **not enough**

# Spherical case

**Theorem [Odaka, appendix to [D 2023], based on [Li 2022]]**

A polarized $G$-spherical manifold $(X, L)$ admits a cscK metric if and only if it is $G$-uniformly K-stable.

**Theorem [D 2023]**

Convex geometric translation of $G$-uniform K-stability for polarized $G$-spherical varieties.

**Theorem [D 2023]**

A rank one polarized $G$-spherical manifold $(X, L)$ admits a cscK metric if and only if it is K-stable with respect to $G$-equivariant **special** test configurations.
The latter translates into a very simple single combinatorial condition.

Later: more on the conjecture.

# Combinatorial condition

$(X, L)$ be a polarized rank one $G$-spherical manifold, with moment polytope $\Delta$.
$\chi \in \Delta$, $\sigma$ generator of $M$ which evaluates non-negatively on the valuation cone,

$$\Delta = \{\chi + t\sigma \mid t \in [s_-, s_+] \subset \mathbb{R}\}$$

Let

$$P(t) = \prod_{\alpha \in \Phi_X^+} \frac{\langle \alpha, \chi + t\sigma \rangle}{\langle \alpha, \varpi \rangle} \qquad Q(t) = \sum_{\alpha \in \Phi_X^+} \frac{\langle \alpha, \varpi \rangle}{\langle \alpha, \chi + t\sigma \rangle} P(t)$$

For a continuous function $g : [s_-, s_+] \to \mathbb{R}$, let

$$\mathcal{L}(g) = g(s_-)P(s_-) + g(s_+)P(s_+) - \int_{s_-}^{s_+} 2g(t)(aP(t) - Q(t))dt$$

where $a$ is the constant such that $\mathcal{L}(1) = 0$.
Then there exists a cscK metric iff

- $\mathcal{L}(\mathrm{id}) > 0$ if $X$ is not horospherical,
- $\mathcal{L}(\mathrm{id}) = 0$ if $X$ is horospherical.

# Test configurations

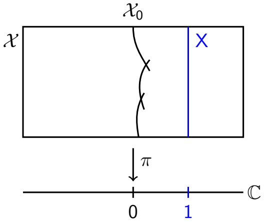
A test configuration for $(X, L)$ consists of the data of

1. a normal variety $\mathcal{X}$,
2. a $\mathbb{C}^*$-action on $\mathcal{X}$,
3. a flat projective, $\mathbb{C}^*$-equivariant morphism $\pi : \mathcal{X} \to \mathbb{C}$,
4. a $\pi$-ample line bundle $\mathcal{L}$ on $\mathcal{X}$,

such that

- $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L^r)$ for some $r \in \mathbb{Z}_{>0}$,

where $(\mathcal{X}_1, \mathcal{L}_1)$ denotes the (scheme-theoretic) fiber of $\pi$ above $1 \in \mathbb{C}$, equipped with the restriction of $\mathcal{L}$.



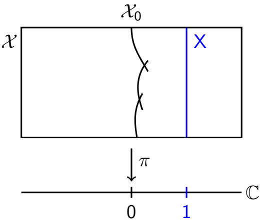
The central fiber $(\mathcal{X}_0, \mathcal{L}_0)$, equipped with an action of $\mathbb{C}^*$, is the important data.

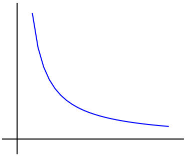
**Upshot:** more symmetries, more singularities

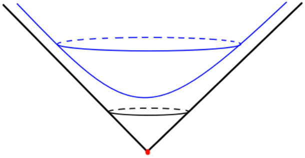
will think of $\mathcal{X}_0$ as a $\mathbb{C}^*$-stable divisor in $\mathcal{X}$.

## Examples

- $\mathbb{P}^1$ degenerates to two intersecting lines (several irreducible components)
  $\mathcal{X} = \{([x : y : z], t); xy - tz^2 = 0\}$
- $\mathbb{P}^1$ degenerates to a double line (non-reduced)
  $\mathcal{X} = \{([x : y : z], t); txy - z^2 = 0\}$
- $\mathbb{P}^1 \times \mathbb{P}^1$ degenerates to a weighted projective space (normal, but singular)





# Special test configurations

**Special test configuration:** A t. c. is *special* if $\mathcal{X}_0$ is normal

**Product test configuration:** A t. c. is *product* if $\mathcal{X}_0$ is isomorphic to $X$.
$\mathbb{C}^*$-action on $(X, L) \Rightarrow$ product test configuration:
$\mathcal{X} = X \times \mathbb{C}$, $\pi$ projection to $\mathbb{C}$, action of $\mathbb{C}^*$ given by $t \cdot (x, s) = (t \cdot x, ts)$
All product test configurations arise from this construction.

**Twist of a test configuration:** $(\mathcal{X}, \mathcal{L})$ be any $G$-equivariant t. c.
$\mu : \mathbb{C}^* \to \mathrm{Aut}^G(X)$ 1psg of $G$-equiv automorphisms of $X$.
Over $\mathbb{C}^*$, family is trivial, can define new $\mathbb{C}^*$-action on $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times \mathbb{C}^*$ by

$$t \cdot (x, s) = (\mu(t) \cdot x, ts)$$

action actually extends to $\mathcal{X}$ and defines a new $G$-equivariant test configuration, the **twist** of $(\mathcal{X}, \mathcal{L})$ by $\mu$.

# Donaldson-Futaki invariant

$$H^0(\mathcal{X}_0, \mathcal{L}_0^k) = \bigoplus_{j=1}^{d_k} V_{j,k} \text{ decomposition in irreducible } \mathbb{C}^*\text{-representations}$$

Each $V_{j,k}$ is of dimension one, and $\mathbb{C}^*$ acts by on it with a weight $\lambda_{j,k}$:

$$z \cdot s = z^{\lambda_{j,k}} s$$

$$\frac{\sum_j \lambda_{j,k}}{k d_k} = F_0 + F_1 \frac{1}{k} + o(\frac{1}{k}) \text{ as } k \to \infty \qquad \text{[Donaldson]}$$

**Donaldson-Futaki invariant** $DF(\mathcal{X}, \mathcal{L}) := -F_1$
**Non-archimedean $J$-functional** $J^{NA}(\mathcal{X}, \mathcal{L}) = \sup\{\frac{\lambda_{j,k}}{k}\} - F_0$
Subtleties:

- $F_0$ depends on the choice of $\mathbb{C}^*$-linearization of $\mathcal{L}$ (but not $F_1$ and $J^{NA}$)
- $DF$ does not vary linearly with base changes $z \mapsto z^m$. Better to work with non-Archimedean Mabuchi functional $M^{NA}$ which is the linear functional which coincide with $DF$ when the central fiber is reduced.

# (Uniform) K-stability

## Definitions

- $(X, L)$ is **K-semistable** if $M^{NA}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations.
- $(X, L)$ is **K-(poly)stable** if $M^{NA}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations, with equality if and only if the test configuration is a product.
- $(X, L)$ is **uniformly K-stable** if there exists a positive constant $\varepsilon > 0$ such that for all test configurations, $M^{NA}(\mathcal{X}, \mathcal{L}) \geq \varepsilon J^{NA}(\mathcal{X}, \mathcal{L})$.
- $(X, L)$ is $G$-**uniformly K-stable** if $\exists \varepsilon > 0$, for all $G$-equivariant test configurations,

$$M^{NA}(\mathcal{X}, \mathcal{L}) \geq \varepsilon \inf_{\mu} J^{NA}(\text{twist of } (\mathcal{X}, \mathcal{L}) \text{ by } \mu)$$

# Test configurations for spherical varieties

$(X, L)$ polarized $G$-spherical variety, $s$ $B$-section of $L$, $\mathrm{div}(s) = \sum_D n_D D$ and $\Delta$ associated polytope (defined by the equations $\rho(D)(m) + n_D \geq 0$)

## Theorem [D]

- $G$-equivariant test configurations of $(X, L)$ are in 1:1 correspondence with negative rational piecewise linear convex functions on the moment polytope $\Delta$, whose slopes are in the opposite valuation cone $-\mathcal{V}$ of $X$.
- special test configurations correspond to linear functions $f \in -\mathcal{V}$
- product test configurations correspond to linear functions $l \in \mathrm{Lin}(\mathcal{V})$
- twists correspond to adding a linear function $l \in \mathrm{Lin}(\mathcal{V})$.

**First key remark:** under the action of $G \times \mathbb{C}^*$, the total space $\mathcal{X}$ is still spherical

# Convex function associated to a test configuration

Assume $\mathcal{X}_0 = \sum D_j$ is reduced (OK up to base change)

$D_j$: prime $G \times \mathbb{C}^*$-stable divisors in the $G \times \mathbb{C}^*$ spherical variety $\mathcal{X}$
$\leftrightarrow$ primitive elements $(u_j, t_j) \in N \times \mathbb{Z}_{<0}$.

Extend $s^r$ to a $B \times \mathbb{C}^*$-equiv section of $\mathcal{L}$ using $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L^r)$ and $\mathbb{C}^*$-action.

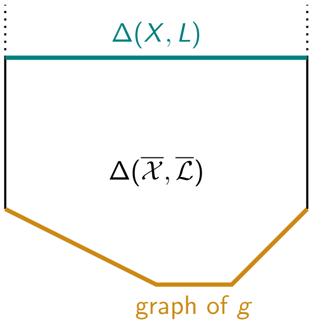
Associated divisor is of the form

$$r\,\overline{\mathrm{div}(s) \times \mathbb{C}^*} + \sum_j n_j D_j$$

The PL function associated to $(\mathcal{X}, \mathcal{L})$ is

$$g : x \mapsto \sup \frac{r u_j(x) + n_j}{t_j}$$

# Picture


$\Delta(X, L)$
$\Delta(\overline{\mathcal{X}}, \overline{\mathcal{L}})$
graph of $g$

# NA functionals

**[Donaldson 2002]:** Using restriction exact sequences, get

$$\sum_{j,k} \lambda_{j,k} = \dim H^0(\mathcal{X}, \mathcal{L}^k) - \dim H^0(X, L^{rk})$$

Toric case: all these dimensions may be computed by counting integral points in dilated convex polytope.

Spherical case: add values of a polynomial (using Weyl dimension formula) at integral points in dilated convex polytope

**[Khovanski-Pukhlikov 1992]**: If the polynomial is homogeneous, defines a valuation on the space of virtual polytopes, in particular, polynomial wrt dilation.

Furthermore, have Minkowski inversion

$$\ominus\Delta = (-1)^{\dim(\Delta)} \operatorname{Int}(-\Delta) = \sum_F (-1)^{\dim(F)}(-F)$$

# Identifying the coefficients

**Toric case:**

$$\mathcal{F}(k) := \#(k\Delta \cap \mathbb{Z}^n) = a_n k^n + a_{n-1} k^{n-1} + \cdots$$

First coeff:

$$a_n = \lim_{k\to\infty} \frac{\# k\Delta \cap \mathbb{Z}^n}{k^n} = \mathrm{Vol}(\Delta)$$

where volume is wrt Lebesgue measure normalized by lattice $\mathbb{Z}^n$.
By Minkowski inversion,

$$\mathcal{F}(-k) = (-1)^n a_n k^n + (-1)^{n-1} a_{n-1} k^{n-1} + \cdots = \#(k\,\mathrm{Int}(\Delta) \cap \mathbb{Z}^n)$$

Second coeff:

$$2a_{n-1} = \lim_{k\to\infty} \frac{\mathcal{F}(k) - (-1)^n \mathcal{F}(-k)}{k^{n-1}} = \mathrm{Vol}(\partial\Delta)$$

**Spherical case:** same idea, get

$$\begin{aligned}\dim H^0(X, L^k) &= \sum_{\lambda \in M \cap k\Delta} \dim V_{k\chi+\lambda} \\ &= k^n \int_\Delta P d\mu + k^{n-1}\left(\frac{1}{2}\int_{\partial\Delta} P d\sigma + \int_\Delta Q d\mu\right) + o(k^{n-1})\end{aligned}$$

*with* $\dim V_{k\lambda} = k^d P(\lambda) + k^{d-1} Q(\lambda) + \cdots$ given by Weyl dimension formula

# NA functionals: conclusion

Set

$$P(x)=\prod_{\alpha\in\Phi_X^+}\frac{\langle x+\chi,\alpha\rangle}{\langle\varpi,\alpha\rangle}\qquad Q(x)=\sum_{\alpha\in\Phi_X^+}\frac{\langle\varpi,\alpha\rangle}{\langle x+\chi,\alpha\rangle}P(x)$$

$$V=\int_\Delta Pd\mu\qquad a=\frac{1}{2V}\left(\int_{\partial\Delta}Pd\sigma+2\int_\Delta Qd\mu\right)$$

## Theorem [D]

Let $f$ be the convex PL function associated to $(\mathcal{X},\mathcal{L})$.

$$M^{NA}(\mathcal{X},\mathcal{L})=\frac{1}{2V}\left(\int_{\partial\Delta}fPd\sigma+\int_\Delta f2(Q-aP)d\mu\right)=:\frac{1}{2V}\mathcal{L}(f)$$

and

$$J^{NA}(\mathcal{X},\mathcal{L})=\frac{1}{V}\int_\Delta(f-\min f)Pd\mu=:\frac{1}{V}\mathcal{J}(f)$$

# Uniform K-stability of polarized spherical varieties

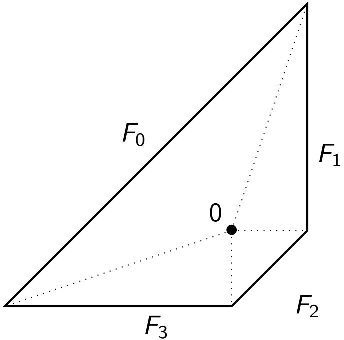
$$\mathcal{L}(f) = \int_{\partial\Delta} fPd\sigma + \int_{\Delta} f2(Q - aP)d\mu$$

$$\mathcal{J}(f) = \int_{\Delta} (f - \min f)Pd\mu$$

$(X, L)$ is $G$-uniformly K-stable if and only if there exists $\varepsilon > 0$ such that for all convex PL function $f$ on $\Delta$ with slopes in $-\mathcal{V}$,

$$\mathcal{L}(f) \geq \varepsilon \inf_{l \in \mathrm{Lin}(\mathcal{V})} \mathcal{J}(f + l)$$

# Towards the conjecture




Choose a point 0 in the interior of $\Delta$, decompose $\Delta$ into pyramids $T_F$ with base the facet $F$ and vertex 0, as the facets $F$ vary.
Let $u_F$ denote the primitive outward normal to the facets of $\Delta$, and let $n_F$ be the numbers such that

$$\Delta = \{x \mid u_F(x) \leq n_F\}$$

# A sufficient condition

Let $(X, L)$ be a polarized $G$-spherical variety

**Theorem**

Assume that for all $F$ and $x \in T_F$,

$$d_x P(x) + (r+1)P(x) + 2n_F(Q - aP)(x) \geq 0$$

then $(X, L)$ is $G$-uniformly K-stable if and only if $(X, L)$ is K-stable with respect to special test configurations.

**Corollary**

Assume that $X$ is smooth. Assume that for all $F$ and $x \in T_F$,

$$d_x P(x) + (r+1)P(x) + 2n_F(Q - aP)(x) \geq 0$$

then there exists a cscK metric in $c_1(L)$ if and only if $(X, L)$ is K-stable with respect to special test configurations.

# Sketch of proof

1. By approximation, can work with smooth convex functions instead of PL convex functions.
2. Normalize functions: work with $f$ smooth convex on $\Delta$, such that $\inf f = 0$ and $d_0 f$ is in a fixed complement subspace $\mathcal{W}$ to $\mathrm{Lin}(\mathcal{V})$. Can always reduce to this by adding an affine function with slope in $\mathrm{Lin}(\mathcal{V})$.
3. On normalized functions, $G$-uniform K-stability writes $\mathcal{L}(f) \geq \epsilon \int_\Delta fPd\mu$
4. Use divergence formula to transform the integral on the boundary to integrals on the interior of the polytope:

$$\int_F fPd\sigma = \frac{1}{n_F}\int_{T_F} \left(P(x)d_xf(x) + rf(x)P(x) + f(x)d_xP(x)\right) d\mu$$

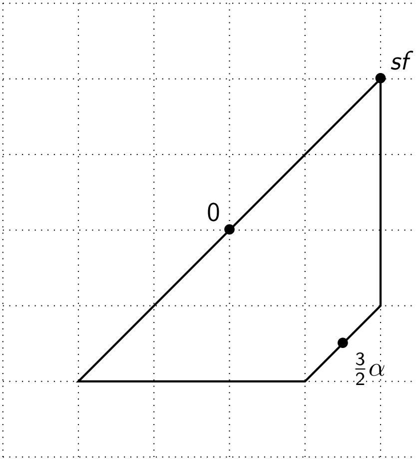
5. Plug into $\mathcal{L}$

$$\begin{aligned}\mathcal{L}(f) = &\sum_F \frac{1}{n_F}\int_{T_F} \left(d_xf(x) - f(x)\right) P(x)d\mu \\ &+ \sum_F \frac{1}{n_F}\int_{T_F} \left((r+1)P(x) + d_xP(x) - 2n_F(aP-Q)(x)\right) f(x)d\mu\end{aligned}$$

# An example

Consider the $\mathrm{SL}_2 \times \mathbb{C}^*$-spherical variety $\mathrm{Bl}_{Q^1} Q^3$. Let $\alpha$ be the unique positive root and $f$ the generating character of $\mathbb{C}^*$. Up to scaling, the moment polytope of an ample line bundle is as on the right

By the sufficient condition, the associated Kähler class admits a cscK metric if $1,683 < s < 3$



# Case of Fano toric manifolds

Recover (by same method) a sufficient condition for properness of (modified) Mabuchi functional on toric manifolds [Zhou-Zhu 2008]

Recall condition: for $x \in T_F$,

$$d_x P(x) + (r+1)P(x) + 2n_F(Q - aP)(x) \geq 0$$

In toric case, $P = 1, Q = 0$, get for any facet $F$

$$r + 1 - 2n_F a \geq 0$$

where $r = \dim(X)$

**Notable particular case:** all $n_F$ are equal
All $n_F = 1$ means the polytope $\Delta$ is *reflexive*
Furthermore, $2a =$ scalar curvature, which in Fano case, for $L = K_X^{-1}$, is $= r$ the dimension
Thus $r + 1 - r = 1 > 0$
The condition varies continuously $\Rightarrow$ condition holds on a neighborhood of $c_1(X)$.

# Case of close to Fano spherical manifolds

$\Delta = \Delta(X, K_X^{-1})$, $X$ spherical, some facets do not have $n_F = 1$.

But when a facet does not have $n_F = 1$, then $P$ actually vanishes on that facet, and the condition associated to that facet may be written differently (can work as if $n_F = 1$).

Can check that the numerical condition is indeed satisfied for $L = K_X^{-1}$, recover the combinatorial criterion for existence of KE metrics on Fano spherical varieties [D 2020]

**Recall conjecture**

On a neighborhood of $c_1(X)$, uniform K-stability is equivalent to K-stability with respect to special test configurations.

[D]: proof for large classes of spherical varieties (open orbit affine with trivial Picard group, toroidal horospherical)
**Difficulties:** The polynomial is non-negative but vanishes... Need to vary the interior point 0 of the polytope as the Kähler class varies to preserve non-negativity.

**Hope for the future:**

1. Geometrical interpretation of the condition
2. proof for all spherical varieties
3. other varieties? What is special about spherical? (e.g. $K_X^{-1}$ big, Mori Dream Space,...)