Effective K-stability of spherical varieties

Talk 1: Introduction to spherical varieties via dimension four

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Motivation : Toric geometry

Definition

A normal n-dimensional algebraic variety X equipped with an effective action of a torus $(\mathbb{C}^*)^n$ is called a toric variety.

1. toric variety $X \leftrightarrow$ fan
2. polarized toric variety $(X^n, L) \leftrightarrow$ integral polytope $\Delta \subset \mathbb{R}^n$
3. Fano toric variety $(X, K_X^{-1}) \leftrightarrow$ reflexive polytope
4. admits a Kähler-Einstein metric $\leftrightarrow \mathrm{Bar}(\Delta) = 0$
5. polarized toric variety (X, L) is K-(semi)stable $\leftrightarrow$ explicit linear functional $\mathcal{L}$ is non-negative on convex functions on Δ

Other group actions

X normal complex algebraic variety
$\circlearrowleft$
G connected complex reductive algebraic group

Basic data : complexity and rank

1. complexity of $G \curvearrowleft X :=$ min codim of an orbit of a Borel subgroup $B \subset G$
2. weight lattice $M :=$ subgroup of weights of rational B-eigenfunctions
3. rank of $G \curvearrowleft X :=$ rank of M

Definition: $G \curvearrowleft X$ *spherical variety* if complexity $= 0$

Theory of spherical varieties: a dictionary, as in the toric case

geometric $\leftrightarrow$ combinatorial / convex

[Luna-Vust 83, Brion, Knop,...]

Recollections on reductive groups: root system

$G \supset B$ Borel subgroup of G, $\quad T \simeq (\mathbb{C}^*)^r$ a maximal torus of B

$X^*(B) = X^*(T) := \{\chi : T \to \mathbb{C}^* \text{ morphism}\} \simeq \mathbb{Z}^r$ group of characters

$\Phi \subset X^*(T)$ root system of (G, T), $\quad \Phi^+ \subset \Phi$ roots of B.

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \forall t \in T, \mathrm{Ad}(t)(x) = \alpha(t)x\}$$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \qquad \mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

Example: GL_n, B upper triangular matrices, T diagonal matrices

$X^*(T)$ generated by $\mathrm{diag}(a_1, \dots, a_n) \mapsto a_j$
Φ is the set of $\alpha_{j,k} : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_j/a_k$ for $j \neq k$, and $\mathfrak{g}_{\alpha_{j,k}} = \mathbb{C}E_{j,k}$
$\alpha_{j,k} \in \Phi^+$ iff $j < k$.

First examples of spherical varieties

1. toric varieties are spherical
2. Bruhat decomposition $\Rightarrow G/B$ spherical e.g. $\mathbb{P}^1 = \mathrm{SL}_2/B$
3. if $B \subset P \subset G$, G/P spherical e.g. $\mathrm{SL}_3 \circlearrowleft \mathbb{P}^2$
4. SL_2 acts on Sym_2 by congruences $\Rightarrow \mathrm{SL}_2 \circlearrowleft \mathbb{P}^2$ spherical
5. diagonal action of SL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ is spherical
6. $\mathrm{SL}_2 \times \mathbb{C}^* \circlearrowleft \mathbb{P}(1, 1, k)$ spherical
7. $\mathrm{SL}_2 \times \mathbb{C}^* \circlearrowleft \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ spherical

These (plus invariant subvarieties) exhaust all 2-dim spherical varieties

Higher dimensions?

Low dimensional spherical Fano manifolds

Dimension 3 Fano spherical: [Hofscheier PhD thesis 2015], not available online

WIP with Pierre-Louis Montagard: classification of spherical Fano fourfolds

Toric varieties:

- dimension 2 well-known
- dimension 3 [Batyrev, Watanabe-Watanabe]
- dimension 4 [Batyrev]
- dimension 5 to 8 by algorithm [Obro]

Only a question about special integral polytopes

Here, first obstacle: determine possible open orbits G/H. Only after, becomes question about special rational polytopes.

Theorem [D.-Montagard, 2023]

Explicit classification of spherical homogeneous spaces of dimension 4.

Strategy for classification of spherical G/H

Standing Assumption: $G = G^{sc} \times (\mathbb{C}^*)^n$ with G^{sc} semisimple, simply connected
$\rightarrow$ action of G^{sc} on G/H has finite central kernel
$\rightarrow$ action of $(\mathbb{C}^*)^n$ on G/H is effective.

Theorem (Brion-Luna-Vust 1986)

Assume BH/H open in G/H. Let $P = \mathrm{Stab}(BH/H)$. There exists a Levi subgroup L of P with connected center C such that $P \cap H = L \cap H$ contains $[L, L]$ and the map

$$P^u \times C/(C \cap H) \rightarrow BH/H, \qquad (p, x) \mapsto p \cdot x$$

is an isomorphism.

In particular,

$$\dim(G/H) = \mathrm{rank}(G/H) + \dim(G/P)$$

and under the standing assumption, P does not contain a simple factor of G.

G	$\mathrm{rank}(G)$	G/P	$\dim(G/P)$
SL_2	1	$\mathbb{P}^1$	1
SL_3	2	$\mathbb{P}^2$	2
SL_2^2	2	$\mathbb{P}^1 \times \mathbb{P}^1$	2
Sp_4	2	Q^3	3
SL_3	2	W	3
Sp_4	2	$\mathbb{P}^3$	3
SL_4	3	$\mathbb{P}^3$	3
$\mathrm{SL}_3 \times \mathrm{SL}_2$	3	$\mathbb{P}^2 \times \mathbb{P}^1$	3
SL_2^3	3	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	3
$\mathrm{SL}_3 \times \mathrm{SL}_2$	3	$W \times \mathbb{P}^1$	4
$\mathrm{Sp}_4 \times \mathrm{SL}_2$	3	$Q^3 \times \mathbb{P}^1$	4
SL_4	3	Q^4	4
$\mathrm{Sp}_4 \times \mathrm{SL}_2$	3	$\mathbb{P}^3 \times \mathbb{P}^1$	4
SL_5	4	$\mathbb{P}^4$	4
$\mathrm{SL}_4 \times \mathrm{SL}_2$	4	$\mathbb{P}^3 \times \mathbb{P}^1$	4
SL_3^2	4	$\mathbb{P}^2 \times \mathbb{P}^2$	4
$\mathrm{SL}_3 \times \mathrm{SL}_2^2$	4	$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$	4
SL_2^4	4	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	4

Recall

$$\dim(G/H) = \operatorname{rank}(G/H) + \dim(G/P)$$

Consequence: strong restriction on possible G

- If $\operatorname{rank}(G/H) = \dim(G/H)$ then $P = G = (\mathbb{C}^*)^n$ and H trivial (toric case).
- If $\operatorname{rank}(G/H) = \dim(G/H) - 1$ then $G^{sc} = \mathrm{SL}_2$.
- If $\operatorname{rank}(G/H) = \dim(G/H) - 2$ then $G^{sc} \in \{\mathrm{SL}_3, \mathrm{SL}_2^2\}$.
- If $\operatorname{rank}(G/H) = \dim(G/H) - 3$ then $G^{sc} \in \{\mathrm{SL}_4, \mathrm{Sp}_4, \mathrm{SL}_3 \times \mathrm{SL}_2, \mathrm{SL}_2^3, \mathrm{SL}_3\}$.
- If $\operatorname{rank}(G/H) = 0$ then $H = P$ (homogeneous case)

If $\dim(G/H) \leq 4$, need only consider those G^{sc}.

Furthermore: if $\operatorname{rank}(G/H) = 1$, spherical homogeneous spaces classified up to parabolic induction.

Parabolic induction

$$G/H = \frac{G \times (G_0/H_0)}{Q}$$

under the action $q \cdot (g, x) = (gq^{-1}, \pi(q) \cdot x)$, for some $\pi : Q \to G_0$ epimorphism from a proper parabolic subgroup Q of G to a connected reductive group G_0

Example: $\mathrm{SL}_2 \curvearrowleft \mathbb{C}^2 \setminus \{0\} = \mathrm{SL}_2 / \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is obtained by parabolic induction:

$$Q = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad \pi : Q \to \mathbb{C}^*, \quad \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \mapsto a$$

Special case: if $G_0 = (\mathbb{C}^*)^r$ ($\Leftrightarrow B^u \subset H$), say G/H is horospherical.

In general, it is characterized by $Q^u \subset H \subset Q$
or even, at the level of Lie algebras $\mathfrak{q}^u \subset \mathfrak{h} \subset \mathfrak{q}$

Next step

Will come back to rank one spherical fourfolds later.

Next goal: classify spherical subgroup H of $G = G^{sc} \times (\mathbb{C}^*)^n$, with $\dim(G/H) \leq 4$ and $G^{sc} \in \{\mathrm{SL}_2, \mathrm{SL}_3, \mathrm{SL}_2^2\}$

1. Algebraic subgroups of SL_2 are well-known
2. Lie subalgebras of $\mathfrak{sl}_3$ and $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ up to conjugation have been classified [Douglas-Repka 2016]

▷ Determine which correspond to spherical subgroups
▷ throw in a torus factor.

Upshot: most spherical homogeneous spaces are obtained by parabolic inductions!

Downside: higher dimension will be a combinatorial nightmare!

Spherical subgroups of $G = \mathrm{SL}_2 \times (\mathbb{C}^*)^n$

(Under standing assumption, up to conjugation and exterior automorphism)

1. $H = \left\langle \begin{bmatrix} \bullet & 0 \\ 0 & \bullet \end{bmatrix} \times \{1\}, \quad \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \pm 1, 1, \dots, 1 \right) \right\rangle$
2. $H = \left\{ \left(\begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix}, a^{-m}, 1 \dots, 1 \right) \right\}$ for some $m \in \mathbb{Z}_{\geq 0}$
3. $H = \left\{ \left(\begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix}, a^{-m}, 1 \dots, 1 \right) \right\}$
4. $H = \left\{ \left(\begin{bmatrix} e^{\frac{2ik\pi}{m}} & b \\ 0 & e^{-\frac{2ik\pi}{m}} \end{bmatrix}, 1 \dots, 1 \right) \right\}$

Spherical subgroups of $G = \mathrm{SL}_3 \times (\mathbb{C}^*)^n$

ϖ_j fundamental weights, χ_j projection to jth $\mathbb{C}^*$ factor, Q maximal parabolic

Assume $\dim G/H \leq 4$ and $\mathrm{Aut}^{G,0}(G/H) \subset G$.

Up to conjugation and G-equivariant automorphism,

1. $n = 2$, $H = \ker(m\varpi_1 + \chi_1 : Q \to \mathbb{C}^*) \cap \ker(\chi_2 : Q \to \mathbb{C}^*)$ for some $m \in \mathbb{Z}_{\geq 0}$,
2. $n = 1$, $H = \ker(m\varpi_1 + \chi_1 : Q \to \mathbb{G}_m)$
3. $n = 1$, $H = \ker(m_1\varpi_1 + m_2\varpi_2 + \chi_1 : B \to \mathbb{G}_m)$
4. $n = 0$, $H = B$
5. $n = 0$, $H = Q$

6. $n = 0$, $H = \langle Q^u, T \rangle$
7. $n = 0$, $H = N(\langle Q^u, T \rangle)$

8. $n = 0$, $H = S(\mathrm{GL}_2 \times \mathrm{GL}_1)$ (only case **not** a parabolic induction)

Spherical subgroups of $G = \mathrm{SL}_2^2 \times (\mathbb{C}^*)^n$

$(\dim(G/H) \leq 4$, up to conjugation and G-equivariant automorphism)

1. G/H is obtained by parabolic induction
2. $n = 0$ and H is one of the following:
    1. $\mathrm{diag}(\mathrm{SL}_2)$
    2. $N(\mathrm{diag}(\mathrm{SL}_2))$
    3. $T_1 \times T_2$
    4. $N(T_1) \times T_2$
    5. $N(T_1) \times N(T_2)$
    6. $\left\langle T_1 \times T_2, \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \right\rangle$
    7. $\mathrm{diag}(B_1)$
    8. $N(\mathrm{diag}(B_1)) = \langle \mathrm{diag}(B_1), (I_2, -I_2) \rangle$
3. $n = 1$ and H is one of the following:
    1. $\mathrm{diag}(\mathrm{SL}_2) \times \{1\}$
    2. $N(\mathrm{diag}(\mathrm{SL}_2)) \times \{1\}$
    3. $\langle \mathrm{diag}(\mathrm{SL}_2), (I_2, -I_2, -1) \rangle$

Parabolic inductions for $G = \mathrm{SL}_2^2 \times (\mathbb{C}^*)^n$

1. $n = 0,\ H = B$
2. $n = 0,\ H = B_1 \times T_2$
3. $n = 0,\ H = B_1 \times N(T_2)$
4. $n = 0,\ H = \ker(m_1\varpi_1 + m_2\varpi_2 : T_1 \times B_2 \to \mathbb{C}^*)$
5. $n = 0,\ H = \left\langle T_1 \times \ker(m_2\varpi_2 : B_2 \to \mathbb{C}^*), \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix} \right) \right\rangle$
6. $n = 1,\ H = \ker(m_1\varpi_1 + m_2\varpi_2 + \chi_1 : B \to \mathbb{C}^*)$
7. $n = 1,\ H = \ker(m_1\varpi_1 + m_2\varpi_2 + \chi_1 : T_1 \times B_2 \times \mathbb{C}^* \to \mathbb{C}^*)$
8. $n = 1,$
   $H = \left\langle T_1 \times \ker(m_2\varpi_2 + \chi_1 : B_2 \times \mathbb{C}^* \to \mathbb{C}^*), \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}, \pm 1 \right) \right\rangle$
9. $n = 2,\ H = \ker(m_1\varpi_1 + \chi_1 : B \to \mathbb{C}^*) \cap \ker(k_1\varpi_1 + k_2\varpi_2 + \chi_2 : B \to \mathbb{C}^*)$

Remaining rank one cases

1. $G = \mathrm{Sp}_4$ and $H = \mathrm{SL}_2 \times \mathrm{Sp}_2$
2. $G = \mathrm{Sp}_4$ and $H = N_{\mathrm{Sp}_4}(\mathrm{SL}_2 \times \mathrm{Sp}_2)$

3. $G = \mathrm{SL}_2^3$ and $H = B_1 \times \mathrm{diag}(\mathrm{SL}_2)$
4. $G = \mathrm{SL}_2^3$ and $H = B_1 \times N_{\mathrm{SL}_2^2}(\mathrm{diag}(\mathrm{SL}_2))$
5. $G = \mathrm{SL}_2^3$ and $H = B_1 \times B_2 \times T_3$
6. $G = \mathrm{SL}_2^3$ and $H = B_1 \times B_2 \times N_{\mathrm{SL}_2}(T_3)$
7. $G = \mathrm{SL}_3 \times \mathrm{SL}_2$ and $H = Q_1 \times T_2$
8. $G = \mathrm{SL}_3 \times \mathrm{SL}_2$ and $H = Q_1 \times N_{\mathrm{SL}_2}(T_2)$

9. $G = \mathrm{SL}_2^3 \times \mathbb{C}^*$ and $H = \ker(m_1\varpi_1 + m_2\varpi_2 + m_3\varpi_3 + \chi : B \to \mathbb{C}^*)$
10. $G = \mathrm{SL}_3 \times \mathrm{SL}_2 \times \mathbb{C}^*$ and $H = \ker(m_1\varpi_1 + m_3\varpi_3 + \chi : Q_1 \times B_2 \times \mathbb{C}^* \to \mathbb{C}^*)$
11. $G = \mathrm{Sp}_4 \times \mathbb{C}^*$ and $H = \ker(m_1\varpi_1 + \chi : Q_{\{\alpha_1\}} \to \mathbb{C}^*)$
12. $G = \mathrm{Sp}_4 \times \mathbb{C}^*$ and $H = \ker(m_2\varpi_2 + \chi : Q_{\{\alpha_2\}} \to \mathbb{C}^*)$
13. $G = \mathrm{SL}_4 \times \mathbb{C}^*$ and $H = \ker(m_1\varpi_1 + \chi : Q_{\{\alpha_1\}} \to \mathbb{C}^*)$

Recollections on reductive groups: representations

Fix $\langle \cdot, \cdot \rangle$ a scalar product on $\mathrm{X}^*(T) \otimes \mathbb{R}$ extending the Killing product.

1. All finite dimensional representations of G are decomposable into direct sums of irreducible representations.
2. There is a bijection between the set of dominant weights $\{\lambda \in \mathrm{X}^*(T) \mid \forall \alpha \in \Phi^+, \langle \alpha, \lambda \rangle \geq 0\}$ and the set of irreducible representations of G up to isomorphism.
3. Explicitely, sending an irreducible representation V to the weight λ of the unique B-eigenvector in V, called the highest weight of V.

We denote by V_λ an irreducible representation with highest weight λ.

Positive Weyl chamber: $(\Phi^+)^\vee := \{\lambda \in \mathrm{X}^*(T) \otimes \mathbb{R} \mid \forall \alpha \in \Phi^+, \langle \alpha, \lambda \rangle \geq 0\}$

$$V = \sum_{\lambda \in \mathrm{X}^*(T) \cap (\Phi^+)^\vee} V_\lambda^{m_\lambda}$$

Recollection on G-varieties: Moment polytope

(X, L) polarized G-variety (i.e. L ample G-linearized line bundle on X)

Moment polytope

$$\Delta = \Delta(X, L) = \mathrm{Conv}\left\{\frac{\lambda}{k}\right\}$$

where $k \in \mathbb{Z}_{>0}$ and λ runs over all characters of B such that there exists a B-eigensection $s \in H^0(X, L^k)$ with eigenvalue λ:

$$\forall b \in B, \quad b \cdot s = \lambda(b)s$$

This is a convex polytope sitting inside the positive Weyl chamber of (G, T, B).

Moment polytope for spherical varieties

X spherical G-variety, L ample G-linearized line bundle on X.

Fix $s \in H^0(X, L^k)$ a B eigensection with weight χ, then [Brion 1989]

$$H^0(X, L^k) = \bigoplus_{\lambda \in k\Delta; \lambda - \chi \in M} V_\lambda$$

in particular, *multiplicity free*

Provides an expression for the dimension of $H^0(X, L^k)$ for tomorrow, thanks to:

Weyl dimension formula

$\dim V_\lambda = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \varpi, \alpha \rangle}{\langle \varpi, \alpha \rangle}$ where $\varpi = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Combinatorial description [Brion 1989]

$\mathrm{div}(s) = \sum n_D D$ sum over B-stable prime divisor, $n_D \in \mathbb{Z}$.

$D \leftrightarrow \mathrm{ord}_D$ valuation on $\mathbb{C}(X) = \mathbb{C}(G/H) \mapsto \rho(D) := \mathrm{ord}_D|_M \in N = \mathrm{Hom}(M, \mathbb{Z})$

$$\Delta - \chi = \{m \in M \otimes \mathbb{R} \mid \rho(D)(m) + n_D \geq 0\}$$

underlying data at the heart of a dictionary similar to that for toric varieties

A *valuation* of $\mathbb{C}(X)$ (the field of rational functions on X) is a group morphism $\nu : \mathbb{C}(X)^* \to \mathbb{R}$ such that $\nu(\mathbb{C}^*) = \{0\}$ and $\nu(f_1 + f_2) \geq \min \nu(f_i)$.

Valuation cone

The *valuation cone* $\mathcal{V} = \mathcal{V}(G/H)$ of X is the image by ρ of the set of G-invariant valuations of $\mathbb{C}(X)$. It is a rational polyedral cone in $N \otimes \mathbb{R}$.

Colors

The set of *colors* $\mathcal{D} = \mathcal{D}(G/H)$ is the set of B-stable prime divisors of G/H.

colored fans $\leftrightarrow$ spherical embeddings $G/H \subset X$

Fano spherical varieties

[Brion 1991] explicit $K_X^{-1} = \mathcal{O}(\sum m_D D)$ with $m_D = 1$ if D is G-stable

Definition

Polytope $Q \subset N \otimes \mathbb{R}$ with vertices $V(Q)$ is *locally factorial G/H-reflexive* if $0 \in \mathrm{Int}(Q)$,

1. $0 \in \mathrm{Int}(Q)$,
2. $\forall D \in \mathcal{D}, \frac{\rho(D)}{m_D} \in Q$,
3. $V(Q) \subset \left((N \cap \mathcal{V}) \cup \{\frac{\rho(D)}{m_D}, D \in \mathcal{D}\}\right)$,
4. $\forall$ facet F st $\mathrm{cone}(F) \cap \mathrm{Int}(\mathcal{V}) \neq \emptyset$, let $\mathcal{D}_F = \{D \in \mathcal{D} \mid \frac{\rho(D)}{m_D} \in F\}$, then
    1. $\rho : \mathcal{D}_F \to \mathrm{cone}(F)$ is injective
    2. $V(F) = \{\frac{\rho(D)}{m_D} \mid D \in \mathcal{D}_F\} \cup \mathcal{C}_F$, and $\mathcal{C}_F \cup \rho(\mathcal{D}_F)$ forms a basis of N

Theorem (Gagliardi-Hofscheier 2015)

X locally factorial Fano iff $(\Delta - \chi)^\vee$ is locally factorial G/H-reflexive

What's left to do?

Recall goal: classification of spherical Fano (locally factorial) fourfolds.

$\Rightarrow$ For each G/H, find all locally factorial G/H-reflexive polytopes.

Rk 1 or 2 rather straightforward **but** rank 3 is more involved!

[Batyrev] toric dim 3 or 4: relies on an understanding of explicit birational geometry of toric manifolds (at least, blowups).
The same understanding is desirable for spherical varieties. Case by case analysis?

Workaround: for rank three, dimension four spherical homogeneous spaces, if one forgets G/H the G/H-reflexive polytopes are smooth / canonical Fano toric polytopes, classified by [Batyrev, Watanabe-Watanabe] / [Kasprzyck]

Still WIP: canonical Fano toric = 674 688 polytopes **up to** $GL_3(\mathbb{Z})$-action
Want to impose as many conditions as possible before going through the list.