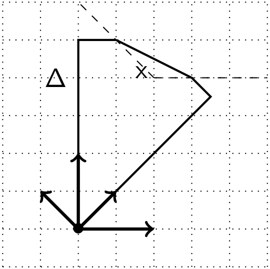
Existence of canonical Kähler metrics on spherical varieties — Lecture 2

Ensemble of Algebra and Geometry



Thibaut Delcroix
Université de Montpellier

Today

Differential geometric aspects of the study of canonical Kähler metrics on horosymmetric manifolds.

- Focus on a subclass of spherical varieties: horosymmetric manifolds
- Proof of KE criterion in that case
- What if there are no KE metrics?
- candidates for alternative canonical Kähler metrics on Fano manifolds
- criterion and proof for these

Hard but fundamental problem

understand the differential geometry of all spherical varieties as well as that of horosymmetric manifolds.

Alternative canonical Kähler metrics on Fano manifolds 1

Recall that $\omega \in c_1(X)$ is KE iff h is constant, where $\mathrm{Ric}(\omega) - \omega = i\,\partial\bar{\partial}\,h$

Kähler-Ricci solitons

X admits a Kähler-Ricci soliton (KRS) if there exists a holomorphic vector field $\xi \in \mathfrak{aut}(X)$ and a Kähler form ω such that

$$\mathrm{Ric}(\omega) - L_\xi \omega = \omega$$

In general, for ξ holomorphic vector field on X, there exists $\theta_{\omega,\xi}$ function st $L_\xi\omega = \sqrt{-1}\partial\bar{\partial}\theta_{\omega,\xi}$ (unique up to additive constant)

Hence ω is a KRS iff $h = \theta_{\omega,\xi}$

Mabuchi metrics

Solutions to the equation

$$\mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}\ln(A\theta_{\omega,\xi} + B) = \omega$$

for some constants A, B, and some holomorphic vector field ξ

Alternative canonical Kähler metrics on Fano manifolds 2

Coupled KE metrics [Witt Nyström+Hultgren]

$\alpha_1, \ldots, \alpha_k$ Kähler classes such that $\alpha_1 + \cdots + \alpha_k = c_1(X)$
$(\omega_1, \ldots, \omega_k) \in \alpha_1 \times \cdots \times \alpha_k$ are coupled KE metrics if

$$\mathrm{Ric}(\omega_1) = \cdots = \mathrm{Ric}(\omega_k) = \omega_1 + \cdots + \omega_k$$

Can focus on the case of two classes: $\mathrm{Ric}(\omega_1) = \mathrm{Ric}(\omega_2) = \omega_1 + \omega_2$.

Coupled multiplier Hermitian structures [D+Hultgren]

$g_1, \ldots, g_k : \mathbb{R} \to \mathbb{R}$ smooth concave functions
$\xi_1, \ldots, \xi_k$ holomorphic vector fields
$(\omega_1, \ldots, \omega_k) \in \alpha_1 \times \cdots \times \alpha_k$ st

$$\begin{aligned} \mathrm{Ric}(\omega_1) - \sqrt{-1}\partial\bar{\partial} g_1(\theta_{\omega_1,\xi_1}) &= \omega_1 + \cdots + \omega_k \\ \cdots &= \cdots \\ \mathrm{Ric}(\omega_k) - \sqrt{-1}\partial\bar{\partial} g_k(\theta_{\omega_k,\xi_k}) &= \omega_1 + \cdots + \omega_k \end{aligned}$$

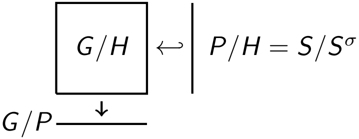
Horosymmetric manifolds

Definition: $X \curvearrowleft G$, X complex manifold, G connected complex linear reductive group, st

- $\exists x \in X$,
- $\exists P$ parabolic subgroup of G, with Levi decomposition $P = SU$
- $\exists \sigma$ group involution of S

and

- $G \cdot x$ dense in X
- $H := \mathrm{Stab}_G(x) \subset P$
- $P/H \simeq S/S^\sigma$ under S-action



More generally, should allow $H \cap S$ with $(H \cap S)^0 = (S^\sigma)^0$, not just S^σ.
Condition on H reads on the Lie algebra: $\mathfrak{h} = \mathfrak{u} \oplus \mathfrak{s}^\sigma$

Examples

1. Generalized flag manifolds G/P: take $\sigma = \mathrm{id}_S$, then $H = P$.

2. Toric manifolds: $G = (\mathbb{C}^*)^n = P = S$, $\sigma : g \mapsto g^{-1}$ group involution since G is abelian, $\{1\} = (S^\sigma)^0$.

   Note: $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{z}(\mathfrak{s})$ and any Lie algebra involution preserves this decomposition. Group level: $S = [S, S]Z(S)$.

3. $[S, S] \subset S^\sigma \longrightarrow$ horospherical manifolds, G/H is a homogeneous fiber bundle in tori $P/H \simeq (\mathbb{C}^*)^r$ over a generalized flag manifold G/P.
   - e.g. $H = \mathrm{Ker}(\chi)$, $\chi : P \to \mathbb{C}^*$ character
   - e.g. $X =$ homogeneous toric bundles
   - e.g. $\mathbb{P}^2$, $\mathrm{Bl}_{1pt}\mathbb{P}^2$ as SL_2-varieties as seen yesterday.

4. $P = G \longrightarrow$ symmetric manifolds
   - e.g. $\mathrm{SL}_n / \mathrm{SO}_n$ $(\sigma(g) = (g^T)^{-1})$
   - e.g. $G = R \times R$, $G/H = R$ $(\sigma(g_1, g_2) = (g_2, g_1))$ group compactifications
   - e.g. $X =$ wonderful compactification [De Concini-Procesi 1983]
   - e.g. variety of complete quadrics

A family of (toric and) horospherical manifolds

$$\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q \curvearrowleft \mathsf{GL}_p \times \mathsf{GL}_q \subset \mathsf{GL}_n$$

induces horospherical structure:

$$\mathbb{P}^{n-1} \curvearrowleft \mathsf{GL}_p \times \mathsf{GL}_q$$

$$\mathrm{Bl}_{\mathbb{P}^{p-1}}(\mathbb{P}^{n-1}) \curvearrowleft \mathsf{GL}_p \times \mathsf{GL}_q$$

$$\mathrm{Bl}_{\mathbb{P}^{p-1},\mathbb{P}^{q-1}}(\mathbb{P}^{n-1}) \curvearrowleft \mathsf{GL}_p \times \mathsf{GL}_q$$

The examples above are all Fano manifolds.
Only the latter is a homogeneous toric bundle.

Important remark

- The above are toric, but the horospherical structure takes into account more information from automorphisms
- In general, a toric manifold is horospherical under the action of a maximal reductive group of automorphisms

A variant: a family of symmetric manifolds

$$\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q \curvearrowleft \mathsf{GL}_p \times \mathsf{GL}_q \subset \mathsf{GL}_n$$

standard quadratic form:

$$\mathcal{Q}_n := \sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{p} z_i^2 + \sum_{i=p+1}^{n} z_i^2 = \mathcal{Q}_p + \mathcal{Q}_q$$

$n-2$-dimensional quadric in $\mathbb{P}^{n-1}$:

$$Q^{n-2} = \{\mathcal{Q}_n = 0\} \curvearrowleft \mathrm{SO}_p \times \mathrm{SO}_q \subset \mathrm{SO}_\mathrm{n}$$

is a symmetric, Fano manifold, as well as:

$$\mathrm{Bl}_{Q^{p-2}}(Q^{n-2}) \curvearrowleft \mathrm{SO}_p \times \mathrm{SO}_q$$

Have $\mathrm{Bl}_{Q^{p-2},Q^{q-2}}(Q^{n-2})$ as well, but it is not Fano

Parabolic induction/homogeneous fiber bundle

General construction

G is a reductive group, P a parabolic subgroup, $\pi : P \to S$ Levi quotient of P, Y S-variety,

$$X := \frac{G \times Y}{P} \text{ where } p \cdot (g, y) = (gp^{-1}, \pi(g) \cdot y)$$

is said to be obtained by parabolic induction from Y. It is a G-homogeneous fiber bundle over G/P with fiber Y.

- Horosymmetric homogeneous spaces are those homogeneous space obtained by parabolic induction from a reductive symmetric space.
- Horospherical homogeneous spaces are those homogeneous space obtained by parabolic induction from tori.

Beware: the word horosymmetric is horo*spherical* glued to symmetric, it is not to symmetric what horospherical is to spherical...

Horosymmetric manifolds as spherical manifolds

G, H, P, U, S, σ data associated to a horosymmetric manifold

Let T_s torus in S, maximal for the property that σ acts on T_s by the inverse.
$T_s \subset T$ σ-stable maximal torus of S

Weight lattice $M = \mathrm{X}^*(T/T \cap H)$

Φ_G resp Φ_S root system of G resp S
Q Borel subgroup opposite to P, Q^u its unipotent radical, Φ_{Q^u} the roots of Q^u
B Borel subgroup such that $T \subset B \subset G$, with corresponding positive roots Φ^+, st $\forall \beta \in \Phi_S^+ := \Phi_S \cap \Phi^+$, either $\sigma(\beta) = \beta$ or $-\sigma(\beta) \in \Phi_S^+$.
Let $\Phi_s := \Phi_S \setminus \Phi_S^\sigma$.

Restricted root system $\bar{\Phi} := \{\bar{\beta} := \beta - \sigma(\beta) \mid \beta \in \Phi_s\} \subset \mathrm{X}^*(T/T \cap H)$

with multiplicities $m_\alpha = \mathrm{Card}\{\beta \in \Phi_s \mid \bar{\beta} = \alpha\}$.
Positive restricted roots: $\bar{\Phi}^+ = \{\bar{\beta} \mid \beta \in \Phi_s^+\}$

Valuation cone $\mathcal{V} = \{v \in N \otimes \mathbb{R} \mid \forall \bar{\beta} \in \bar{\Phi}^+, \alpha(v) \leq 0\}$

Note: weight lattice, restricted root system and valuation cone essentially determined by symmetric fiber $S/S \cap H$ (as expected by parabolic induction)

Color map

Definition

Consider the (finite) set $\mathcal{C}(G/H)$ of B-stable prime divisors of G/H (irreducible components of the complement of the open B-orbit). Identify $\mathcal{C}(G/H)$ with a set of valuations of $\mathbb{C}(G/H)$ (to a function f, associate its order of vanishing along the divisor). The *color map* is the restriction of ρ to $\mathcal{C}(G/H)$, seen as an abstract map from a finite set to $N \otimes \mathbb{R}$.

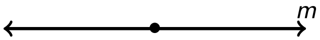
A spherical homogeneous space is fully determined by: weight lattice + valuation cone + color map.

For horosymmetric homogeneous space G/H, with $P = SU$ and $\pi : G/H \to G/P$,

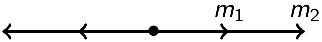
- $\mathcal{C}(G/H)\text{"} = \text{"}\mathcal{C}(S/S \cap H) \cup \mathcal{C}(G/P)$
- For $\mathcal{C}(G/P)$: elements are indexed by the simple roots α that are roots of Q^u and sent to the restriction of the coroot $\alpha^\vee$ to $M \otimes \mathbb{R}$. (the coroot is the unique element in $\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$ st $\forall x \in \mathfrak{t}$, $\alpha(x) = \frac{2\langle x, \alpha^\vee \rangle}{\langle \alpha^\vee, \alpha^\vee \rangle}$)
- For $\mathcal{C}(S/S \cap H)$: the image of the color map is the set of simple restricted coroots. The color map is furthermore injective if the symmetric space is not Hermitian.

Small rank symmetric spaces

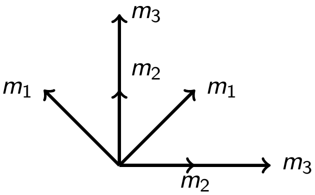
Type A_1



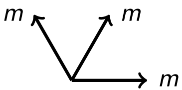
Type BC_1



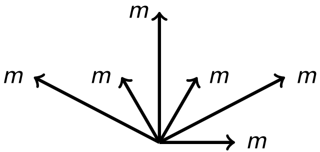
	parameter	$\bar{\Phi}$	multiplicities	Hermitian
$\mathrm{SO}_{m+2}/S(O_1 \times O_{m+1})$	$m \geq 1$	A_1	m	only for $m = 1$
$\mathrm{SL}_{n+1}/S(\mathrm{GL}_1 \times \mathrm{GL}_n)$	$n \geq 2$	BC_1	$(2n-2, 1)$	yes
$\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$	$n \geq 3$	BC_1	$(4n-8, 3)$	no
F_4/B_4		BC_1	$(8, 7)$	no



Type BC_2 or B_2



Type A_2



Type G_2

Parameter	One Representant	$\bar\Phi$	multiplicities	Hermitian?
	$\mathrm{SL}_3 \,/\, \mathrm{SO}_3$	A_2	1	no
	$\mathrm{PGL}_3 \times \mathrm{PGL}_3 \,/\, \mathrm{PGL}_3$	–	2	no
	$\mathrm{SL}_6 \,/\, \mathrm{Sp}_6$	–	4	no
	E_6/F_4	–	8	no
$r \geq 5$	$\mathrm{SL}_r \,/ S(\mathrm{GL}_2 \times \mathrm{GL}_{r-2})$	BC_2	$(2, 2r-8, 1)$	yes
$r \geq 5$	$\mathrm{Sp}_{2r} \,/\, \mathrm{Sp}_4 \times \mathrm{Sp}_{2r-4}$	–	$(4, 4r-16, 3)$	no
	$\mathrm{SO}_{10} \,/\, \mathrm{GL}_5$	–	$(4, 4, 1)$	yes
	$E_6/\, \mathrm{SO}_{10} \times \mathrm{SO}_2$	–	$(6, 8, 1)$	no
$r \geq 5$	$\mathrm{SO}_r \,/ S(O_2 \times O_{r-2})$	B_2	$(1, r-4, 0)$	yes
	$\mathrm{SO}_5 \times \mathrm{SO}_5 \,/\, \mathrm{SO}_5$	–	$(2, 2, 0)$	no
$r = 4$	$\mathrm{Sp}_8 \,/\, \mathrm{Sp}_4 \times \mathrm{Sp}_4$	–	$(3, 4, 0)$	no
	$G_2/\, \mathrm{SO}_4$	G_2	1	no
	$G_2 \times G_2/G_2$	–	2	no

The toric submanifold

Fix θ a Cartan involution of G commuting with σ, and $K = G^\theta$ corresponding maximal compact subgroup. $\mathfrak{a}_s := i\mathfrak{k} \cap \mathfrak{t}_s$ is naturally identified with $\mathrm{X}_*(T_s) \otimes \mathbb{R}$, hence also with $N \otimes \mathbb{R}$ since $T_s \to T/T \cap H$ is an isogeny. Can see $\mathcal{V} \subset \mathfrak{a}_s$.

Key remark: $\mathcal{V}$, identified with $\exp(\mathcal{V})H/H$, is a fundamental domain for the action of K on G/H

The toric submanifold

Consider $Z := \overline{T_s H/H}$. It is a T_s toric submanifold, which intersects any K-orbit in X along an orbit of the semidirect product of the compact subtorus of T_s and the restricted Weyl group.

Upshot: can try to translate everything on Z and use toric geometry / convex geometry!

Kähler metrics / ample line bundles / moment polytopes...

Isotropy character, special polytope

Let L G-linearized ample line bundle on X horosymmetric, fix $\xi \in L_H$.
L is fully determined by the data of:

1. the *isotropy character* $\chi_L : \mathfrak{t} \to \mathbb{C}$ defined by

$$\exp((t + \sigma(t))/2) \cdot \xi = e^{\chi_L(t)} \xi$$

2. the *special divisor* D_s defined as the $\mathbb{Q}$-divisor equal to $\frac{1}{k}$ times the divisor of the B-semi-invariant meromorphic section of L^k whose B-weight vanishes on T_s (which exists and is unique up to constant for large enough k).

To the special (B-stable) divisor $D_s = \sum_D n_D D$ where the sum runs over B-stable prime divisors is associated a convex *special polytope* defined by

$$\Delta_s = \{x \in M \otimes \mathbb{R} \mid \rho(D)(x) + n_D \geq 0\}$$

One recovers the moment polytope Δ of L as $\Delta = \chi_L + \Delta_s$

Conversely, $\Delta_s =$ projection of Δ under $\mathrm{X}^*(T) \otimes \mathbb{R} \to \mathrm{X}^*(T_s) \otimes \mathbb{R}$.

Moment polytope of the restricted line bundle

Define the **toric polytope** Δ^{tor} of L as the moment polytope of $(Z, L|_Z)$.

One can always recover Δ^{tor} from Δ: it is the convex hull of the images of Δ_s under the restricted Weyl group.

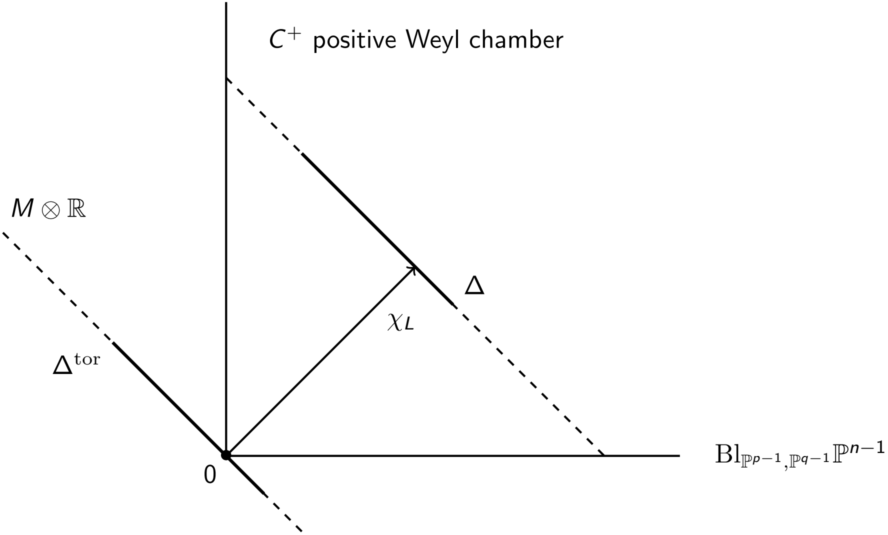
In full generality, one cannot recover L from χ_L and Δ^{tor}. However, it is very often the case that:

$$\Delta_s = \Delta^{\mathrm{tor}} \cap \bar{C}^+$$

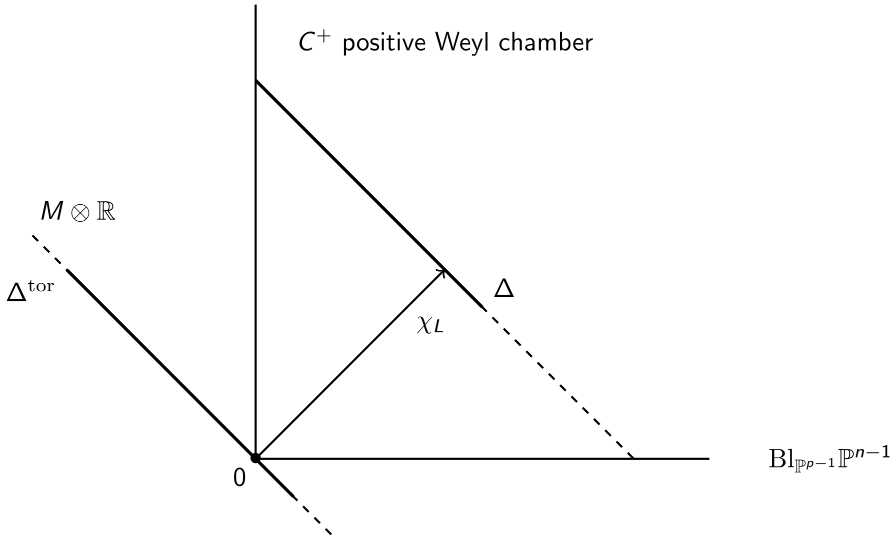
where $\bar{C}^+$ is the positive restricted Weyl chamber in $M \otimes \mathbb{R}$.
For example:

1. it is the case for all line bundles if the symmetric fiber is not Hermitian
2. it is always the case for the anticanonical line bundle
3. in general it is equivalent to $\Delta \cap \mathrm{X}^*(T/T \cap [S,S]) \otimes \mathbb{R} \neq 0$.

Example for horospherical manifolds

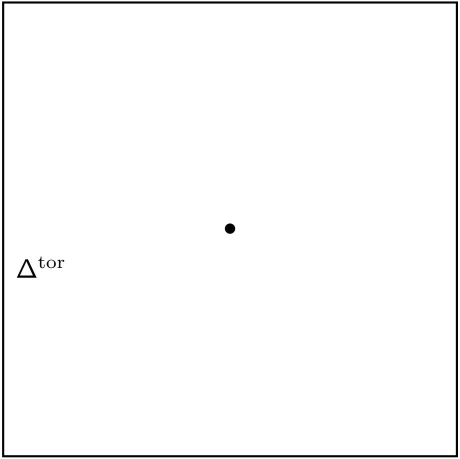


Example for horospherical manifolds



Example for symmetric manifolds

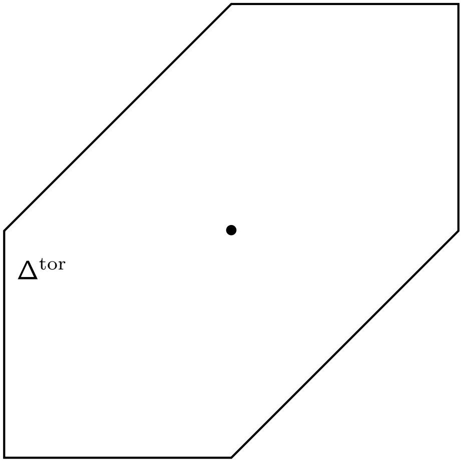
$M \otimes \mathbb{R}$



Q^n

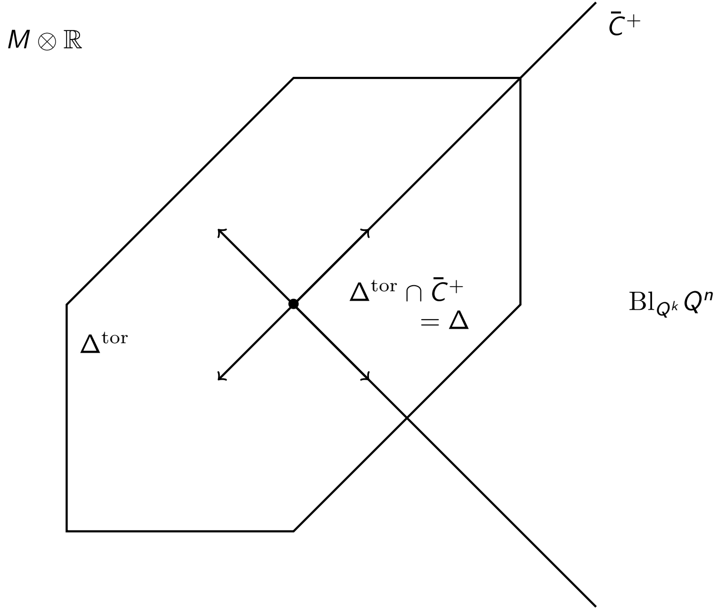
Example for symmetric manifolds

$M \otimes \mathbb{R}$



$\mathrm{Bl}_{Q^k} Q^n$

Example for symmetric manifolds



Discrepancy in Hermitian case

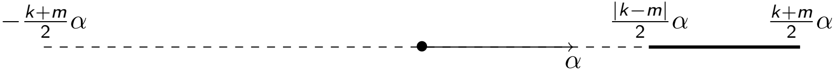
Consider $\mathbb{P}^1 \times \mathbb{P}^1$ equipped with diagonal SL_2 action.

It is a rank 1 Hermitian symmetric manifold! Open orbit is SL_2 / T where T maximal torus (take base point $([1:0],[0:1])$).

The line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ are the $O(k,m)$, ample if $k,\ m > 0$.

Note that Δ^{tor} must be a polytope of the form $[-a,a]$ by invariance under the restricted Weyl group: only one parameter!

More precisely, one can show that the moment polytope and toric polytope for $O(k,m)$ are:



Combinatorial criterion of existence: KE metrics

X horosymmetric Fano $\longrightarrow \Delta_{ac}^{\mathrm{tor}}$ and χ_{ac} for K_X^{-1}

$2\rho_H := \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha - \chi_{ac}$

$$\mathrm{Bar}_{ac} := \int_{\Delta_{ac}^{\mathrm{tor}}\cap\bar{C}^+} p \prod_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\langle\alpha, p+\chi_{ac}\rangle\frac{\mathrm{d}p}{\mathrm{Vol}_{ac}}$$

Theorem [D-Hultgren]

1. X is Kähler-Einstein iff $\mathrm{Bar}_{ac} - 2\rho_H \in \mathrm{Int}((\bar{C}^+)^\vee)$
2. The greatest Ricci lower bound $GRLB(X)$ is
$$\sup\left\{t\in]0,1[\ ;\ 2\rho_H + \frac{t}{1-t}(2\rho_H - \mathrm{Bar}_{ac}) \in \mathrm{Int}(\Delta_{ac}^{\mathrm{tor}} - (\bar{C}^+)^\vee)\right\}$$

Criterion for Kähler-Ricci solitons

$$\mathrm{Bar}_{ac,\xi} := \int_{\Delta_{ac}^{\mathrm{tor}} \cap \bar{C}^+} p e^{\langle \xi, p \rangle} \prod_{\alpha \in \Phi_{Q^u} \cup \Phi_s^+} \langle \alpha, p + \chi_{ac} \rangle \frac{\mathrm{d}p}{\mathrm{Vol}_{ac}}$$

Theorem [D-Hultgren]

1. X admit a KRS iff one can find $\xi \in \mathfrak{X}(T/(T \cap H \cap [G,G])) \otimes \mathbb{R}$ st
$$\mathrm{Bar}_{ac,\xi} - 2\rho_H \in \mathrm{Int}((\bar{C}^+)^\vee)$$
2. The greatest Bakry-Emery-Ricci lower bound for the holom v.f. ξ is
$$\sup \left\{ t \in]0,1[\ ; \ 2\rho_H + \frac{t}{1-t}(2\rho_H - \mathrm{Bar}_{ac,\xi}) \in \mathrm{Int}(\Delta_{ac}^{\mathrm{tor}} - (\bar{C}^+)^\vee) \right\}$$

Note: in all cases, $\exists! \xi$ such that $\mathrm{Bar}_{ac,\xi} - 2\rho_H \in (\bar{C}^+ \cap -\bar{C}^+)^\vee$

Examples

1. For $2 \leq k \leq n-3$, $\mathrm{Bl}_{Q^k}(Q^n)$ does not admit any KRS ($\mathfrak{X}(T/(T \cap [G,G])) \otimes \mathbb{R} = \{0\}$ in this case)
$\mathrm{Bl}_{Q^k}(Q^n)$ has reductive automorphism group, vanishing Futaki invariant, but it is not K-semistable ($GRLB(X) < 1$)
2. Any horospherical Fano manifold admits a KRS

Criterion for coupled Kähler-Einstein metrics

$$\mathrm{Bar}_L := \int_{\Delta_L^{\mathrm{tor}} \cap \bar{C}^+} p \prod_{\alpha \in \Phi_{Q^u} \cup \Phi_s^+} \langle \alpha, p + \chi_L \rangle \frac{\mathrm{d}p}{\mathrm{Vol}_L}$$

Theorem [D-Hultgren]

$\exists$ coupled KEs for $c_1(X) = c_1(L_1) + \cdots + c_1(L_k)$ iff

$$\sum_i \mathrm{Bar}_{L_i} - 2\rho_H \in \mathrm{Int}((\bar{C}^+)^\vee)$$

Examples

1. $\mathrm{Bl}_{\mathbb{P}^1, \mathbb{P}^2}(\mathbb{P}^4)$ admits coupled KEs but no KEs
2. More generally, $\mathrm{Bl}_{\mathbb{P}^{k-1}, \mathbb{P}^k}(\mathbb{P}^{2k+1})$ admits coupled KEs but no KE for k large enough
3. another example: $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-1, 2))$

General statement

$$\mathrm{Bar}_{L,g,\xi} := \int_{\Delta_L^{\mathrm{tor}} \cap \bar{C}^+} p e^{g(\langle \xi, p \rangle)} \prod_{\alpha \in \Phi_{Q^u} \cup \Phi_s^+} \langle \alpha, p + \chi_L \rangle \frac{\mathrm{d}p}{\mathrm{Vol}_L}$$

Theorem [D-Hultgren]

The system of equations

$$\mathrm{Ric}(\omega_j) - \sqrt{-1}\partial\bar{\partial} g_j(\theta_{\omega_j,\xi_j}) = \omega_1 + \cdots + \omega_k + \delta \qquad \forall j$$

admits a solution in $c_1(L_1) \times \cdots \times c_1(L_k)$ iff

$$\sum \mathrm{Bar}_{L_j,g_j,\xi_j} - 2\rho_H \in \mathrm{Int}\left(\Delta_{[\delta]}^{\mathrm{tor}} + (\bar{C}^+)^\vee\right)$$

introduce a smooth semi-positive $(1,1)$-form δ in the right hand side
$\longrightarrow$ allow *twisted* canonical metrics
(moment polytope $\Delta_{[\delta]}^{\mathrm{tor}}$ makes sense as well)

General setting for old and recent results

Recover for example:

1. [Wang+Zhu 2004] KRS on toric manifolds
2. [Podestà+Spiro 2010] KRS on homogeneous toric bundles
3. [Chi Li 2011] greatest Ricci lower bound on toric manifolds
4. [D 2017] KE metrics and greatest Ricci lower bound on group compactifications
5. [Yi Yao 2017] greatest Ricci lower bound on homogeneous toric bundles
6. [Hultgren 2017] coupled KRS on toric manifolds
7. [Li+Zhou 2017] Mabuchi metrics on group compactifications

Interpolating between KE and KRS

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Definition

1. $k \in \mathbb{Z}_{\geq 0}$, call ω a Mab^k-metric if $\exists A, B, \xi$,

$$\mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}\ln\left((A\theta_{\omega,\xi}+B)^k\right) = \omega$$

2. $P(X) := \inf\{k \in \mathbb{Z}_{\geq 0}; X \text{ admits a } \mathrm{Mab}^k\text{-metric}\}$

Open Question: $P(X)$ is finite iff X admits a KRS?
(e.g. for toric and horospherical manifolds)
Does it reflect other (algebro-)geometric information on the manifold?

Examples: $P\left(\mathbb{P}^1\right) = 0$, $\quad P\left(\mathrm{Bl}_{\mathbb{P}^0}\mathbb{P}^2\right) = 1$, $\quad P\left(\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))\right) = 2$,
$P\left(\mathrm{Bl}_{Q^3}(Q^2)\right) = +\infty$

Other open questions

1. $\exists$? Fano manifold with reductive $\mathrm{Aut}(X)$ and no coupled KE metrics?
   Conjecturally: $\mathrm{Bl}_{Q^k}(Q^n)$ How to prove such a non-exitence result?
2. $\exists$? non KE Fano horosymmetric manifold X with $GRLB(X) = 1$? spherical?
3. Alternative canonical Kähler metrics on Fano manifolds with unipotent $\mathrm{Aut}(X)$?
4. $\exists$ Fano 3fold X with no KRS, reductive $\mathrm{Aut}(X)$, $GRLB(X) \neq 1$?
   a Fano 4fold? horosymmetric? spherical?
   for 5fold, $\mathrm{Bl}_{Q^2}(Q^5)$ is an example
   there are no horosymmetric Fano 3fold example
5. $\exists$? Picard rank one Fano manifold with no KRS, not K-semistable?
6. $\exists$? Fano manifold with KRS and irrational soliton vector field ξ?
7. Does $\mathrm{Bl}_{\mathbb{P}^k,\mathbb{P}^{n-k-1}}\mathbb{P}^n$ always admit coupled KEs?

Proof

- **Step 1:** Reduce to C^0 estimates
- a variation on Yau's proof and [Hultgren 2017] for coupled KRS
- valid for all Fano manifolds
- (but assumption of concavity of g_i added here)

- **Step 2:** Translate the equations into real Monge-Ampère equations
- for horosymetric Fano manifolds
- follows essentially from [D., Crelle 2019]

- **Step 3:** Prove C^0 estimates
- for more general real Monge-Ampère equations on cones
- wide generalization of [Wang-Zhu 2004, D. 2017, Hultgren 2017]

Step 1: reduction to C^0 estimates

- Use the following continuity method

$$\mathrm{Ric}(\omega_{j,t}) - \sqrt{-1}\partial\bar{\partial}g_j(\theta_{\omega_{j,t},\xi_j}) = t\sum_l \omega_{l,t} + (1-t)\sum_l \omega_{l,\mathrm{ref}} + \delta \qquad \forall j$$

- adapt arguments of Yau to this setting
- get

Theorem [D-Hultgren]

Assume a priori C^0-estimates hold on normalized potentials of $\omega_{l,s} - \omega_{l,\mathrm{ref}}$ for $s \in [0,t] \subset [0,1]$, then there exists a solution for all $s \in [0,t]$.

- Note that X horosymmetric is not required here

Step 2: $\mathbb{C}$MA eqns on horosymmetric manifolds

Toric potential

To a K-invariant Hermitian metric h on K_X^{-1} (generally L) associate: its toric potential $u : \mathfrak{a}_s \to \mathbb{R}$ defined by

$$u(a) = -2 \ln|\exp(a) \cdot \xi|$$

for some fixed $\xi \in L_H$.

- Recall that $\exp(\mathcal{V})H/H$ is a fundamental domain for the action of K, so $u|_{\mathcal{V}}$ fully determines the Hermitian metric, and u is invariant under the restricted Weyl group action.
- If h is positively curved, then u is a (strictly) convex function
- Furthermore, $\{d_a u \mid a \in \mathfrak{a}_s\} = \mathrm{Int}(-2\Delta^{\mathrm{tor}})$ (follows from toric case).

Monge-Ampère operator

At $\exp(a)H/H$, have

$$\frac{\omega_h^n}{n!} = \det(d_a^2 u)\frac{\prod_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\langle\alpha, 2\chi - d_a u\rangle}{\prod_{\alpha\in\Phi_{Q^u}} e^{-2\alpha(a)}\prod_{\beta\in\Phi_s^+}\sinh(-2\beta(a))}|\exp(a)\cdot\xi_{ac}\wedge\overline{\exp(a)\cdot\xi_{ac}}|$$

In fact, compute ω_h itself in coordinates, using the local coordinates:

$$g\exp(\sum_j z_j l_j + \sum_{\alpha\in\Phi_{Q^u}} z_\alpha e_\alpha + \sum_{\beta\in\Phi_s^+} z_\beta \tau_\beta)H$$

near gH, where e_α root vector and $\tau_\beta = e_\beta - \sigma(e_\beta)$.
To compute, reduce to a function: if G/H symmetric non-Hermitian, then $L|_{G/H}$ is trivial, there is a global potential ϕ for the curvature ω_h: $\omega_h|_{G/H} = i\,\partial\bar\partial\,\phi$.
Compute $\frac{\partial^2}{\partial z_{\beta_1}\partial \bar{z_{\beta_2}}}\Big|_0 \phi(\exp(a)\exp(z_{\beta_1}e_{\tau_1} + z_{\beta_2}e_{\tau_2})$ etc, by using: K-invariance, Baker-Campbell-Hausdorff formula, Lie algebra bracket computations...

If parabolic induction, pullback to G via $\pi : G \to G/H$, work with the *quasipotential* $\phi : G \to \mathbb{R}, g \mapsto -2\ln|g \cdot \xi|$ which has equivariance properties with respect to H involving the isotropy character, then $\pi^*\omega_h = i\,\partial\bar{\partial}\,\phi$

If symmetric fiber is Hermitian, the computation works, but get a more complicated expression unless $L|_{S/S\cap H}$ is trivial.

$\Rightarrow$ can express the Kähler forms and their Ricci forms in terms of convex function!

Up to considering an additional torus factor in G, and making the action effective, any holomorphic vector field ξ commuting with the action of G is induced by the action of the center of G $\xi \in \mathfrak{z}(\mathfrak{g}) \subset \mathfrak{t}$, θ_{ξ,ω_h} is K-invariant, determined by $\theta_{\xi,\omega_h}(\exp(a)H/H) = -d_a u(\xi)$ for $a \in \mathfrak{a}_s$.

Step 3: C^0 estimates for $\mathbb{R}$MA eqns

We actually derive C^0 estimates for a larger family of (paths of) systems of $\mathbb{R}$MA equations on some convex polyedral cone $C \subset \mathbb{R}^r$:

$$\det(d^2 u_{i,t}) G_i(du_{i,t}) = J \prod_{l=1}^{k} e^{-tu_{l,t}-(1-t)u_{j,\mathrm{ref}}} \quad \text{on } C \subset \mathbb{R}^r$$

where

- unknown $u_{i,t}$ are smooth convex functions on $\mathbb{R}^r$ st $\overline{\{d_x u_{i,t} \mid x \in C\}} = \Delta_i \subset (\mathbb{R}^r)^*$ fixed convex polytopes
- G_i continuous functions on Δ_i, smooth and positive on $\mathrm{Int}(\Delta_i)$, $\int_{\Delta_i} G_i = 1$, $G_i^{-\epsilon}$ integrable for some $\epsilon > 0$
- J continuous, positive on $\mathrm{Int}(C)$, vanishing on ∂C
- $j = -\ln J$ is smooth and convex on $\mathrm{Int}(C)$
- its recession function $j_\infty : \xi \in C \mapsto \lim_{t\to\infty} j(x+t\xi)/t$ satisfies some technical conditions

Crucial condition

$$\det(d^2 u_{i,t}) G_i(du_{i,t}) = J \prod_{l=1}^{k} e^{-tu_{l,t}-(1-t)u_{j,\mathrm{ref}}} \quad \text{on } C \subset \mathbb{R}^r$$

Let $\Delta =$ Minkowski sum of all Δ_i, v_Δ support function of Δ,
$\mathrm{Bar}_i = \int_{\Delta_i} pG_i(p)dp \in (\mathbb{R}^r)^*$

Condition $(\dagger_t)$

$$\begin{aligned}(t\textstyle\sum_i \mathrm{Bar}_i + (1-t)v_\Delta + j_\infty)(\xi) &\geq 0 \text{ for } \xi \in C,\\ &= 0 \text{ iff } t = 1, -\xi \in C, j_\infty(-\xi) = -j_\infty(\xi).\end{aligned}$$

Theorem [D+Hultgren]

Let $t_0 > 0$ and $t \in (t_0, 1]$.

1. If $(\dagger_t)$ is true then there are C^0 estimates on $[t_0, t]$
2. If $(\dagger_t)$ is not true then there are no smooth solutions at t.

Basic idea

Consider the simple equation on $\mathbb{R}$

$$g(u'(t))u''(t) = \sinh(t)e^{-u(t)}$$

Assume that u is even, strictly convex and $u(t) - a|t| = O(1)$, then $u'(\mathbb{R}) =]-a, a[$ and $u'(\mathbb{R}_+^*) =]0, a[$.
Multiply the equation bu $u'(t)$, integrate over $\mathbb{R}_+^*$, use change of variable $p = u'(t)$.
Get

$$\int_0^a pg(p)dp = \int_{\mathbb{R}_+^*} u'(t)\sinh(t)e^{-u(t)}$$

Write $j(t) = -\ln\sinh(t)$, then $\int_{\mathbb{R}_+^*}(j' + u')e^{-j-u} = 0$, hence

$$\int_0^a pg(p)dp + \int_{\mathbb{R}_+^*} j'(t)\sinh(t)e^{-u(t)} = 0$$

But $j' \leq -1$ hence

$$\int_0^a pg(p)dp - 1 \leq 0$$