Mabuchi's K-energy functional on horosymmetric varieties

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Positivity Concepts on Holomorphic Line Bundles and Theories on Canonical Kähler Metrics, January 29

Introduction

X^n projective complex manifold

$\mathcal{L}$ ample line bundle on X

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Question:

Given an explicit polarized Kähler manifold $(X, \mathcal{L})$, how can we determine if it admits a cscK metric ?

Recall

h Hermitian metric on $\mathcal{L}$,
$z = (z_j)$ local holomorphic coordinates,
s local trivialization

local potential: $\phi(z) := -\ln |s(z)|_h$
curvature form: $\omega = i\partial\bar{\partial}\phi$, Kähler iff $(\partial^2\phi/\partial z_j \partial \bar{z}_k) > 0$
Ricci curvature: $\mathrm{Ric}(\omega) = i\partial\bar{\partial}(-\ln \det \partial^2\phi/\partial z_j \partial \bar{z}_k)$
scalar curvature: $S = n\,\mathrm{Ric}(\omega) \wedge \omega^{n-1}/\omega^n$

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Constant scalar curvature equation:

$$S = \bar{S} = \frac{n\,K_X^{-1}\cdot\mathcal{L}^{n-1}}{\mathcal{L}^n}$$

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But: very few sufficient conditions of existence

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Theorem [Chen, Cheng, arXiv:1801.05907]

A projective polarized manifold $(X, \mathcal{L})$ admits a cscK metric in $c_1(\mathcal{L})$

if and only if

the Mabuchi functional is coercive (modulo automorphisms) in the space of Kähler metrics in $c_1(\mathcal{L})$.

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Note that it is not yet

Yau-Tian-Donaldson conjecture

Existence of cscK equivalent to (uniform ?) K-stability

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Even with K-stability (i.e. assuming YTD),
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Notable exception: $\mathcal{L} = K_X^{-1}$ Fano case, where K-stability with respect to equivariant special test configurations is enough [Datar-Szekelyhidi], e.g.

Theorem [D. 2016]

Combinatorial caracterization of Kähler-Einstein spherical manifolds.

Remark: all the varieties I will mention in this talk are spherical.

Example

Note: On a projective homogeneous manifold G/P under a linear semisimple complex group G, any Kähler class admits a cscK metric.

e.g. $\mathrm{Gr}(r, m) =$ Grassmannian of r-vector subspaces in $\mathbb{C}^m$.

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The manifold X_m

Consider, for $m \geq 5$, the manifold with diagonal action

$$Y := \mathrm{Gr}(2, m) \times \mathrm{Gr}(m-2, m) \curvearrowleft \mathrm{SL}_m(\mathbb{C})$$

3 orbits:

- $Y_c := \{(V_1^2, V_2^{m-2}); V_1 \subset V_2\}$ closed orbit (partial flag manifold)
- $\{\dim(V_1 \cap V_2) = 1\}$ codimension one orbit
- $\{\dim(V_1 \cap V_2) = 0\}$ open dense orbit

Define X_m as the blow up of Y at Y_c.

Statement for X_5

Let D_1 denote the exceptional divisor in X_m and let D_2 denote the other G-invariant divisor.

Theorem [D.]

Consider the line bundle $\mathcal{L} = \mathcal{O}(k_1 D_1 + k_2 D_2)$ on X_5. Then it admits a cscK metric provided

$$c < \frac{k_1}{k_2} - 1 < C$$

where $c \simeq 0,31$ and $C \simeq 0,54$.

Remarks

- $\mathcal{O}(k_1 D_1 + k_2 D_2)$ on X_m is ample if and only if $0 < \frac{k_1}{k_2} - 1 < 1$.
- $\mathfrak{aut}(X_m) = \mathrm{sl}_m(\mathbb{C})$: X_m is not homogeneous under a bigger group.
- In particular: infinitely many new classes with cscK metrics.
- this manifold is Fano Kähler-Einstein, and $K_{X_5}^{-1} = \mathcal{O}(7D_1 + 5D_2)$ $(7/5 - 1 = 0.4)$.
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- Same sort of statement true for X_m for all the values of m I tested, but *a priori* have to test independently for all m.
- Despite previous remarks, this is not (only) an openness statement.
- This is not an Arezzo-Pacard type theorem using the blowup structure (e.g. because classes are far from the boundary of the Kähler cone).

Symmetric spaces

G complex connected linear reductive group (e.g. $\mathrm{SL}_m(\mathbb{C})$)
σ complex group involution of G

Symmetric space

H closed subgroup of G. The homogeneous space G/H is called a symmetric space if $\exists \sigma$ s. t.

$$\mathfrak{h} = \mathfrak{g}^\sigma.$$

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Three types for $\mathrm{SL}_m(\mathbb{C})$

- Type AI: SL_m / SO_m,
- Type AII: $\mathrm{SL}_{2m} / \mathrm{Sp}_{2m}$,
- Type AIII: $\mathrm{SL}_m / S(\mathrm{GL}_r \times \mathrm{GL}_{m-r})$.

Restricted root system

T_s torus in G, maximal for the property that $\sigma(t) = t^{-1}$ for all $t \in T_s$.

$T_s \subset T$ σ-stable maximal torus in G.

R root system of (G, T)

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Restricted root system

$$\bar{R} := \{\alpha - \sigma(\alpha); \alpha \in R \setminus R^\sigma\}$$

is a (possibly non-reduced) root sytem.
Correspondingly: restricted Weyl group $\bar{W}$, positive restricted Weyl chamber $\bar{C}^+$, etc.

Multiplicity of a restricted root $\alpha \in \bar{R}$:

$$m_\alpha = \mathrm{Card}\{\beta \in R; \beta - \sigma(\beta) = \alpha\}$$

Wonderful compactifications

Assume $H = N_G(G^\sigma)$.

$r := \dim(T_s)$ rank of G/H

Theorem [De Concini, Procesi, 1983]

There exists a unique smooth G-equivariant compactification X of G/H, which satisfies the following additional properties:

- $X \setminus (G/H) = \bigcup_{i=1}^r D_i$ is a union of smooth divisors with simple normal crossing D_i,
- orbits of G in X are in bijection with subsets $I \subset \{1, \ldots, r\}$, via: closure of orbit $= \bigcap_{i \in I} D_i$.

Ample line bundles

Any line bundle on X which admits a meromorphic G-equivariant section writes

$$\mathcal{L} = \mathcal{O}(k_1 D_1 + \cdots + k_r D_r) \to X$$

e.g.

$$K_X^{-1} = \mathcal{O}(k_1^{ac} D_1 + \cdots + k_r^{ac} D_r)$$

May number the divisors and simple restricted roots so that:

Theorem ([De Concini Procesi] or [Brion])

The line bundle $\mathcal{L}$ is ample if and only if $k_1\alpha_1 + \cdots + k_r\alpha_r$ is in the positive restricted Weyl chamber.

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Note: there may be more l.b. if G/H is Hermitian.

Associated Polytopes

$$\Delta = \mathrm{Conv}(w \cdot \sum_{j=1}^{r} k_j \alpha_j; w \in \bar{W}\} \qquad \Delta^+ = \Delta \cap \bar{C}^+$$

$$\Delta^+ = \sum_{j=1}^{r} \tilde{\Delta}_j^+$$

decomposition by cones over facets not contained in restricted Weyl walls.

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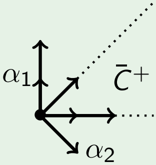
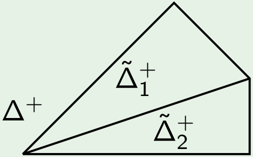
decomposition by cones over facets not contained in restricted Weyl walls.

Type AIII, $r = 2$ (our initial example X_m)

restricted root system is BC_2, $\mathcal{L}$ is ample iff $0 < \frac{k_1}{k_2} - 1 < 1$.





Main Theorem for wonderful compactifications

$$P_{DH}(q) := \prod_{\alpha \in \bar{R}^+} \kappa(\alpha, q)^{m_\alpha} \qquad \kappa \ \text{Killing form}$$

Theorem [D.]

Assume P_{DH} vanishes at least quadratically on restricted Weyl walls. Then the line bundle $\mathcal{O}(k_1 D_1 + \cdots + k_r D_r)$ admits a cscK metric if

$$\min_{j=1}^{r} \left((n+1) \frac{k_j^{ac}}{k_j} - \bar{S} \right) > 0$$

and

$$\sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} \left((n+1) \frac{k_j^{ac}}{k_j} - \bar{S} \right) \left(\min_{j=1}^{r} \frac{k_j^{ac}}{k_j} q - \sum_{\alpha \in R^+ \setminus R^\sigma} \alpha \right) P_{DH}(q) dq \in \mathrm{Relint}(\mathrm{Cone}(\bar{R}^+))$$

Invariant metrics

X wonderful compactification of G/H. $\mathcal{L} = \mathcal{O}(\sum k_j D_j)$.
Fix K maximal compact subgroup in G (e.g. $\mathrm{SU}_m \subset \mathrm{SL}_m$)

$$\mathbb{R}^r \simeq \mathfrak{a}_s := \mathfrak{t}_s \cap i\mathfrak{k}$$

Proposition [D.] (*toric potentials*)

A positively curved K-invariant metric h on $\mathcal{L}$ is encoded by a smooth strictly convex function u on $\mathbb{R}^r$, invariant under the action of $\bar{W}$:

$$u(a) = -\ln|\exp(a) \cdot \xi|_h$$

(for some choice $\xi \in \mathcal{L}_{eH}$.)
Furthermore,

$$\{d_a u, a \in \mathfrak{a}_s\} = \Delta$$

Curvature form

$$\mathfrak{g}/\mathfrak{h} = \mathfrak{t}_s \bigoplus_{\alpha \in R^+ \setminus R^\sigma} \mathbb{C}(e_\alpha - \sigma(e_\alpha))$$

Theorem [D.]

Let h invariant metric and ω its curvature form. For $a \in \mathfrak{a}_s$ outside restricted Weyl walls and in appropriate coordinates,

$$\omega = \begin{pmatrix} d^2u & & & 0 \\ & \frac{\kappa(\alpha^{(1)}, d_a u)}{\sinh(2\alpha^{(1)}(a))} & & \\ & & \dots & \\ 0 & & & \frac{\kappa(\alpha^{(l)}, d_a u)}{\sinh(2\alpha^{(l)}(a))} \end{pmatrix}$$

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! It is not of this form on a line bundle with no meromorphic G-equivariant section !

Consequences

- Monge-Ampère operator

$$\omega^n = \frac{P_{DH}(d_a u)}{J(a)} \det(d_a^2 u) dV$$

where dV is a G-invariant volume form on G/H.

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- Ricci curvature, Scalar curvature
- Integration formula using Legendre transform

$$p = d_a u,$$
$$u^*(p) = p \cdot a - u(a),$$
$$\int_X \psi(a)\omega^n = \int_{\Delta^+} \psi(d_p u^*) P_{DH}(p) dp$$

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- Mabuchi functional (details to come)

Recall Mabuchi functional

$\mathcal{H}$ = space of Kähler forms in $c_1(\mathcal{L})$.

Mabuchi functional

$$\mathcal{M}(\omega) = \int_0^1 \int_X \dot{\phi}_t(\bar{S} - S_t)\omega_t^n dt$$

where $(\omega_t = \omega_{\mathrm{ref}} + i\partial\bar{\partial}\phi_t)_t$ smooth path between a fixed ω_{ref} and ω.

Euler-Lagrange equation = Constant scalar curvature equation.

Coercivity

Distance on $\mathcal{H}$

There exists a notion of distance d_1 on $\mathcal{H}$, satisfying

$$\left| d_1(\omega_{\mathrm{ref}}, \omega) - \int_0^1 \int_X \dot{\phi}_t (\omega_{\mathrm{ref}}^n - \omega_t^n) dt \right| < C$$

Coercivity

$\mathcal{M}$ is coercive if there exists $\epsilon, C > 0$ such that for all $\omega \in \mathcal{H}$,

$$\mathcal{M}(\omega) > \epsilon \; d_1(\omega_{\mathrm{ref}}, \omega) - C$$

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Coercivity (modulo G-action)

$\mathcal{M}$ is coercive if there exists $\epsilon, C > 0$ such that for all $\omega \in \mathcal{H}$,

$$\mathcal{M}(\omega) > \epsilon \inf_{g \in G} d_1(\omega_{\mathrm{ref}}, g \cdot \omega) - C$$

Mabuchi functional on wonderful compactifications

$$\begin{aligned}\mathcal{M}(\omega) = &\sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} (nu^*(p) + d_p u^*(p)) P_{DH}(p) dp \\ &- \bar{S} \int_{\Delta^+} u^*(p) P_{DH}(p) dp \\ &- \int_{\Delta^+} \ln(J(d_p u^*)) P_{DH}(p) dp \\ &- \int_{\Delta^+} \ln(\det(u_{ij}^*)) P_{DH}(p) dp\end{aligned}$$

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Mabuchi functional on wonderful compactifications

$$\begin{aligned}\mathcal{M}(\omega) = &\ \mathcal{M}^{\ell}(u^*) \\ &- \int_{\Delta^+} (\ln(J(d_p u^*)) - \sum_{\alpha \in R^+ \setminus R^\sigma} d_p u^*(\alpha)) P_{DH}(p) dp \\ &- \int_{\Delta^+} \ln(\det(u^*_{ij})) P_{DH}(p) dp\end{aligned}$$

Mabuchi functional on wonderful compactifications

$$\begin{aligned}\mathcal{M}(\omega) = {} & \mathcal{M}^{\ell}(u^*) \\ & + \mathcal{M}^{n\ell}(u^*)\end{aligned}$$

Strategy of proof, from toric case

Normalization

Using automorphisms, may assume u^* normalized i.e. $\inf u^* = u^*(0) = 0$

In toric case, Donaldson proves:

$$\mathcal{M}^\ell > \epsilon \int_{\partial\Delta^+} u^* d\sigma$$

$$\Downarrow$$

$$\mathcal{M} \text{ is coercive}$$

The analogous result is much harder in our case so we introduce simplifying assumptions (P_{DH} vanishes quadratically)

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The analogous result is much harder in our case so we introduce simplifying assumptions (P_{DH} vanishes quadratically)
Then need to prove

$$\mathcal{M}^\ell > \epsilon \int_{\partial\Delta^+} u^* d\sigma$$

Method from [Zhou, Zhu, 2008]

Method to prove estimate linear part in the toric case.

Assume there is a sequence of normalized u_j^* such that

$$\int_{\partial\Delta^+} u_j^* d\sigma = 1 \qquad \mathcal{M}^\ell(u_j^*) \to 0$$

By a compactness result of Donaldson, u_j^* converges to a convex function, uniformly on compact subsets of the interior of Δ^+.

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Write $\mathcal{M}^\ell(u_j^*)$ as a sum of positive terms,
get contradiction.

$$F_j := \left((n+1)\frac{k_j^{ac}}{k_j} - \bar{S}\right)$$

$$p_0 := \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} pF_j P_{DH}(p)dp \Big/ \int_{\Delta^+} P_{DH}(p)dp$$

$$\begin{aligned}
\mathcal{M}^\ell \geq & \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} F_j(u^*(p) - u^*(p_0) - d_{p_0}u^*(p - p_0))P_{DH} \\
& + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} d_p u^* \left(\frac{k_j^{ac}}{k_j} p_0 - \sum_{\alpha \in R^+ \setminus R^\sigma} \alpha \right) P_{DH} \\
& + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} F_j d_{p_0} u^*(p - p_0) P_{DH} \\
& + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} \left(F_j - \frac{k_j^{ac}}{k_j} \right) u^*(p_0) P_{DH}
\end{aligned}$$

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& + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} d_p u^* \left(\frac{k_j^{ac}}{k_j} p_0 - \sum_{\alpha \in R^+ \setminus R^\sigma} \alpha \right) P_{DH} \geq 0 \\
& + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} F_j d_{p_0} u^*(p-p_0) P_{DH} = 0 \\
& + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} \left(F_j - \frac{k_j^{ac}}{k_j} \right) u^*(p_0) P_{DH} = 0
\end{aligned}$$

Horosymmetric varieties: definition

Definition

Assume $X \curvearrowleft G$ is almost homogeneous with open dense orbit G/H. Then X is *horosymmetric* if there exists

- a parabolic subgroup P of G, with Levi decomposition $P = LP^u$,
- a complex involution σ of L,

such that

$$\mathrm{Lie}(H) = \mathrm{Lie}(P^u) \oplus \mathrm{Lie}(L^\sigma)$$

As consequence, G/H is a homogeneous fibration over the generalized flag manifold G/P with fiber the symmetric space $L/L \cap H$.
The fibration may extend to the whole of X or not.

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The fibration may extend to the whole of X or not.
Say X is *toroidal* if the fibration does extend, with fiber a compactification of $L/L \cap H$ which dominates the wonderful compactification of $L/N_L(L \cap H)$.

More general Theorem

Theorem [D.]

Combinatorial sufficient criterion for existence of cscK metrics on polarized manifolds $(X, \mathcal{L})$ where:

- X is a *toroidal* horosymmetric variety, with Duistermaat-Heckman polynomial vanishing at least quadratically on restricted walls,
- $\mathcal{L}$ is an ample line bundle, which restricts to a trivial line bundle on $L/L \cap H$, and satisfies the numerical condition.

Version for pairs: caracterization of log-KE metrics on such manifolds. also coercivity of log-Mabuchi functional for different polarizations.

Horosymmetric varieties: examples I

Trivial fiber: generalized flag manifolds.
Trivial base: wonderful compactifications of symmetric spaces, but also

Biequivariant group compactifications

$X \curvearrowleft G \times G$ with $x \in X$ such that $\overline{G \times G \cdot x} = X$ and $\mathrm{Stab}_{G\times G}(x) = \mathrm{diag}(G)$.
A version of the coercivity criterion is due to [Li, Zhou, Zhu] (there have subsequent on Mabuchi metrics [Li, Zhou] and Sasaki-Einstein metrics [Li, Zhu]).

Horosymmetric varieties: examples II

Toric manifolds

[Wang Zhu 2004] A Fano toric manifold (X, K_X^{-1}) with associated moment polytope Δ is Kähler-Einstein if and only if the barycenter of Δ is the origin.

The condition is closed: indeed, [Zhou Zhu 2008] gives not an open condition but shows K-stable wrt special equivariant t.c. equivalent to coercivity on the range of l.b. considered!

In fact they have an open condition for coercivity of modified Mabuchi functional related to existence of extremal metrics (Weiyong He arXiv:1801.07636 for extremal version of Chen-Cheng).

Horosymmetric varieties: examples III

Non-trivial fibration:

Homogeneous toric bundles

This is a G-manifold X equipped with a G-equivariant map $X \to G/P$ to a generalized flag manifold, such that the action of P on the fiber Z factorizes by a torus $(\mathbb{C}^*)^r$ and the fiber is toric for this action.

Horospherical varieties

When the fibration structure above does not extend to the whole variety.

Horosymmetric varieties: examples IV

Remarkable Property

A G-stable hypersurface in a G-horosymmetric variety is also a G-horosymmetric variety.

Closure of orbits in wonderful compactifications

They are the model of (toroidal) horosymmetric varieties:
Given $I \subset \{1, \dots r\}$ there exists a parabolic subgroup P_I of G, with σ-stable Levi subgroup L_I, such that the closure of the orbit corresponding to I is an equivariant fibration over G/P_I with fiber the wonderful compactification of $L'_I/(L')^\sigma_I$.
(L'_I is the derived subgroup of L_I)

Thank you!

Reference: ArXiv:1712.00221