Mabuchi’s K-energy functional on horosymmetric varieties

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Introduction

General statement for wonderful compactifications

An analogue of Guillemin-Abreu-Donaldson

Proof of coercivity criterion

Horosymmetric varieties
Introduction

\( X^n \) projective complex manifold

\( \mathcal{L} \) ample line bundle on \( X \)
Introduction

\( X^n \) projective complex manifold

\( \mathcal{L} \) ample line bundle on \( X \)

**Question:**

Given an explicit polarized Kähler manifold \((X, \mathcal{L})\), how can we determine if it admits a cscK metric?
Recall

$h$ Hermitian metric on $\mathcal{L}$,
$z = (z_j)$ local holomorphic coordinates,
$s$ local trivialization

local potential: $\phi(z) := - \ln |s(z)|_h$

curvature form: $\omega = i \partial \bar{\partial} \phi$, Kähler iff $(\partial^2 \phi / \partial z_j \partial \bar{z}_k) > 0$

Ricci curvature: $\operatorname{Ric}(\omega) = i \partial \bar{\partial}(- \ln \det \partial^2 \phi / \partial z_j \partial \bar{z}_k)$

scalar curvature: $S = n \operatorname{Ric}(\omega) \wedge \omega^{n-1}/\omega^n$
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scalar curvature: $S = n \text{Ric}(\omega) \wedge \omega^{n-1} / \omega^n$

Constant scalar curvature equation:

$$S = \bar{S} = \frac{n K_X^{-1} \cdot \mathcal{L}^{n-1}}{\mathcal{L}^n}$$
Classical obstructions: Matsushima’s Theorem, K-stability
But: very few sufficient conditions of existence

Key recent result: Chen and Cheng’s solution to Tian’s Conjecture:

Theorem [Chen, Cheng, arXiv:1801.05907]
A projective polarized manifold \((X, L)\) admits a cscK metric in \(c_1(L)\) if and only if the Mabuchi functional is coercive (modulo automorphisms) in the space of Kähler metrics in \(c_1(L)\).

Note that it is not yet Yau-Tian-Donaldson conjecture
Existence of cscK equivalent to (uniform ?) K-stability
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**Yau-Tian-Donaldson conjecture**

Existence of cscK equivalent to (uniform ?) K-stability
How can we use these results?

Even with K-stability (i.e. assuming YTD),
even for toric surfaces (where Donaldson proved YTD),
\emph{a priori} infinite number of conditions to check...
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Even with K-stability (i.e. assuming YTD), even for toric surfaces (where Donaldson proved YTD), \textit{a priori} infinite number of conditions to check...

Notable exception: $\mathcal{L} = K_X^{-1}$ Fano case, where K-stability with respect to equivariant special test configurations is enough [Datar-Szekelyhidi], e.g.

\textbf{Theorem [D. 2016]}

Combinatorial caracterization of Kähler-Einstein spherical manifolds.

Remark: all the varieties I will mention in this talk are spherical.
Example

Note: On a projective homogeneous manifold $G/P$ under a linear semisimple complex group $G$, any Kähler class admits a cscK metric.

e.g. $\text{Gr}(r, m) = \text{Grassmannian of } r\text{-vector subspaces in } \mathbb{C}^m$. 
Example

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The manifold $X_m$

Consider, for $m \geq 5$, the manifold with diagonal action

$$Y := \text{Gr}(2, m) \times \text{Gr}(m - 2, m) \cap \text{SL}_m(\mathbb{C})$$

3 orbits:

- $Y_c := \{(V_1^2, V_2^{m-2}); V_1 \subset V_2\}$ closed orbit (partial flag manifold)
- $\{\dim(V_1 \cap V_2) = 1\}$ codimension one orbit
- $\{\dim(V_1 \cap V_2) = 0\}$ open dense orbit

Define $X_m$ as the blow up of $Y$ at $Y_c$. 
Statement for $X_5$

Let $D_1$ denote the exceptional divisor in $X_m$ and let $D_2$ denote the other $G$-invariant divisor.

Theorem [D.]

Consider the line bundle $\mathcal{L} = \mathcal{O}(k_1D_1 + k_2D_2)$ on $X_5$. Then it admits a cscK metric provided

$$c < \frac{k_1}{k_2} - 1 < C$$

where $c \simeq 0, 31$ and $C \simeq 0, 54$. 
Remarks

- $\mathcal{O}(k_1D_1 + k_2D_2)$ on $X_m$ is ample if and only if $0 < \frac{k_1}{k_2} - 1 < 1$.
- $\text{aut}(X_m) = \text{sl}_m(\mathbb{C})$: $X_m$ is not homogeneous under a bigger group.
- In particular: infinitely many new classes with cscK metrics.
- this manifold is Fano Kähler-Einstein, and $K_{X_5}^{-1} = \mathcal{O}(7D_1 + 5D_2)$ ($7/5 - 1 = 0.4$).
- Same sort of statement true for $X_m$ for all the values of $m$ I tested, but a priori have to test independently for all $m$. 
Remarks

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- Despite previous remarks, this is not (only) an openness statement.

- This is not an Arezzo-Pacard type theorem using the blowup structure (e.g. because classes are far from the boundary of the Kähler cone).
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5 Horosymmetric varieties
Symmetric spaces

$G$ complex connected linear reductive group (e.g. $\text{SL}_m(\mathbb{C})$)

$\sigma$ complex group involution of $G$

Symmetric space

$H$ closed subgroup of $G$. The homogeneous space $G/H$ is called a symmetric space if $\exists \sigma$ s. t.

$$\mathfrak{h} = \mathfrak{g}^\sigma.$$

Example: open $\text{SL}_m$-orbit in $\text{Gr}(r, m) \times \text{Gr}(m - r, m)$
Symmetric spaces

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$$h = g^\sigma.$$ 

Example: open $\text{SL}_m$-orbit in $\text{Gr}(r, m) \times \text{Gr}(m - r, m)$

Three types for $\text{SL}_m(\mathbb{C})$

- Type AI: $\text{SL}_m / \text{SO}_m$,
- Type AII: $\text{SL}_{2m} / \text{Sp}_{2m}$,
- Type AIII: $\text{SL}_m / S(\text{GL}_r \times \text{GL}_{m-r})$. 
Restricted root system

$T_s$ torus in $G$, maximal for the property that $\sigma(t) = t^{-1}$ for all $t \in T_s$.

$T_s \subset T$ $\sigma$-stable maximal torus in $G$.

$R$ root system of $(G, T)$
Restricted root system

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$T_s \subset T$ $\sigma$-stable maximal torus in $G$.

$R$ root system of $(G, T)$

**Restricted root system**

\[ \tilde{R} := \{ \alpha - \sigma(\alpha); \, \alpha \in R \setminus R^\sigma \} \]

is a (possibly non-reduced) root system.
Correspondingly: restricted Weyl group $\tilde{W}$, positive restricted Weyl chamber $\tilde{C}^+$, etc.

Multiplicity of a restricted root $\alpha \in \tilde{R}$:

\[ m_\alpha = \text{Card}\{ \beta \in R; \, \beta - \sigma(\beta) = \alpha \} \]
Wonderful compactifications

Assume $H = N_G(G^\sigma)$.

$r := \dim(T_s)$ rank of $G/H$

**Theorem [De Concini, Procesi, 1983]**

There exists a unique smooth $G$-equivariant compactification $X$ of $G/H$, which satisfies the following additional properties:

- $X \setminus (G/H) = \bigcup_{i=1}^{r} D_i$ is a union of smooth divisors with simple normal crossing $D_i$,

- orbits of $G$ in $X$ are in bijection with subsets $I \subset \{1, \ldots, r\}$, via: closure of orbit $= \bigcap_{i \in I} D_i$. 

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Ample line bundles

Any line bundle on $X$ which admits a meromorphic $G$-equivariant section writes

$$L = \mathcal{O}(k_1D_1 + \cdots + k_rD_r) \to X$$

e.g.

$$K_X^{-1} = \mathcal{O}(k_{1}^{ac}D_1 + \cdots + k_{r}^{ac}D_r)$$

May number the divisors and simple restricted roots so that:

**Theorem ([De Concini Procesi] or [Brion])**

The line bundle $L$ is ample if and only if $k_1\alpha_1 + \cdots + k_r\alpha_r$ is in the positive restricted Weyl chamber.
Ample line bundles

Any line bundle on $X$ which admits a meromorphic $G$-equivariant section writes

$$\mathcal{L} = \mathcal{O}(k_1 D_1 + \cdots + k_r D_r) \to X$$

e.g.

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May number the divisors and simple restricted roots so that:

**Theorem ([De Concini Procesi] or [Brion])**

The line bundle $\mathcal{L}$ is ample if and only if $k_1 \alpha_1 + \cdots + k_r \alpha_r$ is in the positive restricted Weyl chamber.

Note: there may be more l.b. if $G/H$ is Hermitian.
Associated Polytopes

\[ \Delta = \text{Conv}(w \cdot \sum_{j=1}^{r} k_j \alpha_j; \ w \in \bar{W}) \quad \Delta^+ = \Delta \cap \tilde{C}^+ \]

\[ \Delta^+ = \sum_{j=1}^{r} \tilde{\Delta}_j^+ \]

decomposition by cones over facets not contained in restricted Weyl walls.
Associated Polytopes

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\[ \Delta^+ = \sum_{j=1}^{r} \tilde{\Delta}_j^+ \]

decomposition by cones over facets not contained in restricted Weyl walls.

Type AIII, \( r = 2 \) (our initial example \( X_m \))

restricted root system is \( BC_2 \), \( \mathcal{L} \) is ample iff \( 0 < \frac{k_1}{k_2} - 1 < 1 \).
Main Theorem for wonderful compactifications

\[ P_{DH}(q) := \prod_{\alpha \in \bar{R}^+} \kappa(\alpha, q)^{m_\alpha} \]

\( \kappa \) Killing form

**Theorem [D.]**

Assume \( P_{DH} \) vanishes at least quadratically on restricted Weyl walls. Then the line bundle \( \mathcal{O}(k_1 D_1 + \cdots + k_r D_r) \) admits a cscK metric if

\[
\min_{j=1}^r \left( (n+1) \frac{k_j^{ac}}{k_j} - \bar{S} \right) > 0
\]

and

\[
\sum_{j=1}^r \int_{\tilde{\Delta}_j^+} \left( (n+1) \frac{k_j^{ac}}{k_j} - \bar{S} \right) \left( \min_{j=1}^r \frac{k_j^{ac}}{k_j} - q - \sum_{\alpha \in \bar{R}^+ \setminus R^\sigma} \alpha \right) P_{DH}(q) dq \\
\in \text{Relint}(\text{Cone}(\bar{R}^+))
\]
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Invariant metrics

X wonderful compactification of $G/H$. $\mathcal{L} = \mathcal{O}(\sum k_j D_j)$. Fix $K$ maximal compact subgroup in $G$ (e.g. $SU_m \subset SL_m$)

$$\mathbb{R}^r \cong a_s := t_s \cap i\mathfrak{k}$$

**Proposition [D.] (toric potentials)**

A positively curved $K$-invariant metric $h$ on $\mathcal{L}$ is encoded by a smooth strictly convex function $u$ on $\mathbb{R}^r$, invariant under the action of $\tilde{\mathbb{W}}$:

$$u(a) = -\ln |\exp(a) \cdot \xi|_h$$

(for some choice $\xi \in \mathcal{L}_{eH}$.) Furthermore,

$$\{d_a u, a \in a_s\} = \Delta$$
Curvature form

\[ g/h = t_s \bigoplus_{\alpha \in R^+ \setminus R^\sigma} \mathbb{C}(e_\alpha - \sigma(e_\alpha)) \]

**Theorem [D.]**

Let \( h \) invariant metric and \( \omega \) its curvature form. For \( a \in \mathfrak{a}_s \) outside restricted Weyl walls and in appropriate coordinates,

\[
\omega = \begin{pmatrix}
  d^2 u & 0 \\
  \frac{\kappa(\alpha^{(1)}, d_u a)}{\sinh(2\alpha^{(1)}(a))} & \cdots \\
  0 & \frac{\kappa(\alpha^{(l)}, d_u a)}{\sinh(2\alpha^{(l)}(a))}
\end{pmatrix}
\]
Curvature form

\[ \mathfrak{g}/\mathfrak{h} = t_s \bigoplus_{\alpha \in R^+ \setminus R^\sigma} \mathbb{C}(e_{\alpha} - \sigma(e_{\alpha})) \]

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Let \( h \) invariant metric and \( \omega \) its curvature form. For \( a \in \mathfrak{a}_s \) outside restricted Weyl walls and in appropriate coordinates,

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\omega = \begin{pmatrix}
  d^2 u & 0 \\
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  0 & \frac{\kappa(\alpha^{(l)}_a, d_a u)}{\sinh(2\alpha^{(l)}_a(a))}
\end{pmatrix}
\]

! It is not of this form on a line bundle with no meromorphic \( G \)-equivariant section !
Consequences

- Monge-Ampère operator

\[ \omega^n = \frac{P_{DH}(d_a u)}{J(a)} \det(d_a^2 u) dV \]

where \(dV\) is a \(G\)-invariant volume form on \(G/H\).
Consequences

- Monge-Ampère operator

\[ \omega^n = \frac{P_{DH}(du)}{J(a)} \det(d^2u) dV \]

where \( dV \) is a \( G \)-invariant volume form on \( G/H \).

- Ricci curvature, Scalar curvature
Consequences

- Monge-Ampère operator
  \[ \omega^n = \frac{P_{DH}(d_a u)}{J(a)} \det(d_a^2 u) dV \]
  where \( dV \) is a \( G \)-invariant volume form on \( G/H \).

- Ricci curvature, Scalar curvature

- Integration formula using Legendre transform
  \[ p = d_a u, \]
  \[ u^*(p) = p \cdot a - u(a), \]
  \[ \int_X \psi(a) \omega^n = \int_{\Delta^+} \psi(d_p u^*) P_{DH}(p) dp \]
Consequences

- Monge-Ampère operator

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where \( dV \) is a \( G \)-invariant volume form on \( G/H \).

- Ricci curvature, Scalar curvature

- Integration formula using Legendre transform

\[ p = d_a u, \]
\[ u^*(p) = p \cdot a - u(a), \]
\[ \int_X \psi(a) \omega^n = \int_{\Delta^+} \psi(dp u^*) P_{DH}(p) dp \]

- Mabuchi functional (details to come)
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Recall Mabuchi functional

\[ M(\omega) = \int_0^1 \int_X \phi_t (\bar{S} - S_t) \omega_t^n dt \]

where \( (\omega_t = \omega_{\text{ref}} + i \partial \bar{\partial} \phi_t)_t \) smooth path between a fixed \( \omega_{\text{ref}} \) and \( \omega \).

Euler-Lagrange equation = Constant scalar curvature equation.

\[ \mathcal{H} = \text{space of Kähler forms in } c_1(\mathcal{L}). \]
Coercivity

Distance on $\mathcal{H}$

There exists a notion of distance $d_1$ on $\mathcal{H}$, satisfying

$$\left| d_1(\omega_{\text{ref}}, \omega) - \int_0^1 \int_X \phi_t(\omega_{\text{ref}}\n - \omega^n_t)dt \right| < C$$

Coercivity

$\mathcal{M}$ is coercive if there exists $\epsilon, C > 0$ such that for all $\omega \in \mathcal{H}$,

$$\mathcal{M}(\omega) > \epsilon \ d_1(\omega_{\text{ref}}, \omega) - C$$
Coercivity

Distance on $\mathcal{H}$

There exists a notion of distance $d_1$ on $\mathcal{H}$, satisfying

$$|d_1(\omega_{\text{ref}}, \omega) - \int_0^1 \int_X \dot{\phi}_t (\omega_{\text{ref}}^n - \omega_t^n) dt| < C$$

Coercivity (modulo $G$-action)

$\mathcal{M}$ is coercive if there exists $\epsilon, C > 0$ such that for all $\omega \in \mathcal{H}$,

$$\mathcal{M}(\omega) > \epsilon \inf_{g \in G} d_1(\omega_{\text{ref}}, g \cdot \omega) - C$$
Mabuchi functional on wonderful compactifications

\[ \mathcal{M}(\omega) = \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} (nu^*(p) + d_p u^*(p)) P_{DH}(p) dp \]

\[ - \mathcal{S} \int_{\Delta^+} u^*(p) P_{DH}(p) dp \]

\[ - \int_{\Delta^+} \ln(J(d_p u^*)) P_{DH}(p) dp \]

\[ - \int_{\Delta^+} \ln(\det(u^*_{ij})) P_{DH}(p) dp \]
Mabuchi functional on wonderful compactifications

\[ M(\omega) = \sum_{j=1}^{r} \int_{\tilde{\Delta}^+_j} (nu^*(p) + d_p u^*(p))P_{DH}(p)dp \]

\[- \tilde{S} \int_{\Delta^+} u^*(p)P_{DH}(p)dp \]

\[- \int_{\Delta^+} \sum_{\alpha \in R^+ \setminus R^\sigma} d_p u^*(\alpha)P_{DH}(p)dp \]

\[- \int_{\Delta^+} (\ln(J(d_p u^*))) - \sum_{\alpha \in R^+ \setminus R^\sigma} d_p u^*(\alpha))P_{DH}(p)dp \]

\[- \int_{\Delta^+} \ln(\det(u^*_{ij}))P_{DH}(p)dp \]
Mabuchi functional on wonderful compactifications

$$M(\omega) = M^\ell(u^*)$$

$$- \int_{\Delta^+} \left( \ln(J(d_p u^*)) \right) - \sum_{\alpha \in R^+ \setminus R^\sigma} d_p u^*(\alpha) P_{DH}(p) dp$$

$$- \int_{\Delta^+} \ln(\det(u^*_{ij})) P_{DH}(p) dp$$
Mabuchi functional on wonderful compactifications

\[ \mathcal{M}(\omega) = \mathcal{M}^\ell(u^*) + \mathcal{M}^{n\ell}(u^*) \]
Strategy of proof, from toric case

Normalization

Using automorphisms, may assume $u^*$ normalized i.e. $\inf u^* = u^*(0) = 0$

In toric case, Donaldson proves:

$$ M^\ell > \epsilon \int_{\partial \Delta^+} u^* d\sigma $$

\[ \Downarrow \]

$M$ is coercive

The analogous result is much harder in our case so we introduce simplifying assumptions ($P_{DH}$ vanishes quadratically)
Strategy of proof, from toric case

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Then need to prove

$$M^\ell > \epsilon \int_{\partial \Delta^+} u^* d\sigma$$
Method from [Zhou, Zhu, 2008]

Method to prove estimate linear part in the toric case.

Assume there is a sequence of normalized $u_j^*$ such that

$$\int_{\partial \Delta^+} u_j^* \, d\sigma = 1 \quad M^\ell(u_j^*) \to 0$$

By a compactness result of Donaldson, $u_j^*$ converges to a convex function, uniformly on compact subsets of the interior of $\Delta^+$. 
Method from [Zhou, Zhu, 2008]

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Write $M^\ell(u_j^*)$ as a sum of positive terms,
Method from [Zhou, Zhu, 2008]

Method to prove estimate linear part in the toric case.

Assume there is a sequence of normalized $u_j^*$ such that

$$
\int_{\partial \Delta^+} u_j^* d\sigma = 1 \quad \mathcal{M}^\ell(u_j^*) \to 0
$$

By a compactness result of Donaldson, $u_j^*$ converges to a convex function, uniformly on compact subsets of the interior of $\Delta^+$.

Write $\mathcal{M}^\ell(u_j^*)$ as a sum of positive terms, get contradiction.
\[ F_j := \left( (n + 1) \frac{k_j^{ac}}{k_j} - \tilde{S} \right) \]

\[ p_0 := \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} p F_j P_{DH}(p) dp / \int_{\Delta^+} P_{DH}(p) dp \]

\[ \mathcal{M}^\ell \geq \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} F_j (u^*(p) - u^*(p_0) - d_{p_0} u^*(p - p_0)) P_{DH} \]

\[ + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} d_p u^* \left( \frac{k_j^{ac}}{k_j} p_0 - \sum_{\alpha \in R^+ \setminus R^\sigma} \alpha \right) P_{DH} \]

\[ + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} F_j d_{p_0} u^*(p - p_0) P_{DH} \]

\[ + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} \left( F_j - \frac{k_j^{ac}}{k_j} \right) u^*(p_0) P_{DH} \]
\[ F_j := \left( (n + 1) \frac{k_j^{ac}}{k_j} - \bar{S} \right) \]

\[ p_0 := \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} p F_j P_{DH}(p) dp / \int_{\Delta^+} P_{DH}(p) dp \]

\[ \mathcal{M}^\ell \geq \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} F_j (u^*(p) - u^*(p_0) - d_{p_0} u^*(p - p_0)) P_{DH} \geq 0 \]

\[ + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} d_p u^* \left( \frac{k_j^{ac}}{k_j} p_0 - \sum_{\alpha \in R^+ \setminus R^\sigma} \alpha \right) P_{DH} \geq 0 \]

\[ + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} F_j d_{p_0} u^*(p - p_0) P_{DH} = 0 \]

\[ + \sum_{j=1}^{r} \int_{\tilde{\Delta}_j^+} \left( F_j - \frac{k_j^{ac}}{k_j} \right) u^*(p_0) P_{DH} = 0 \]
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Horosymmetric varieties: definition

**Definition**
Assume \( X \curvearrowright G \) is almost homogeneous with open dense orbit \( G/H \). Then \( X \) is *horosymmetric* if there exists

- a parabolic subgroup \( P \) of \( G \), with Levi decomposition \( P = LP^u \),
- a complex involution \( \sigma \) of \( L \),

such that

\[
\text{Lie}(H) = \text{Lie}(P^u) \oplus \text{Lie}(L^\sigma)
\]

As consequence, \( G/H \) is a homogeneous fibration over the generalized flag manifold \( G/P \) with fiber the symmetric space \( L/L \cap H \). The fibration may extend to the whole of \( X \) or not.
Horosymmetric varieties: definition

**Definition**

Assume $X \lhd G$ is almost homogeneous with open dense orbit $G/H$. Then $X$ is *horosymmetric* if there exists

- a parabolic subgroup $P$ of $G$, with Levi decomposition $P = LP^u$,
- a complex involution $\sigma$ of $L$,

such that

$$\text{Lie}(H) = \text{Lie}(P^u) \oplus \text{Lie}(L^\sigma)$$

As consequence, $G/H$ is a homogeneous fibration over the generalized flag manifold $G/P$ with fiber the symmetric space $L/L \cap H$.

The fibration may extend to the whole of $X$ or not.

Say $X$ is *toroidal* if the fibration does extend, with fiber a compactification of $L/L \cap H$ which dominates the wonderful compactification of $L/N_L(L \cap H)$. 

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Mabuchi on Horosymmetric

Osaka
More general Theorem

**Theorem [D.]**

Combinatorial sufficient criterion for existence of cscK metrics on polarized manifolds \((X, \mathcal{L})\) where:

- \(X\) is a *toroidal* horosymmetric variety, with Duistermaat-Heckman polynomial vanishing at least quadratically on restricted walls,
- \(\mathcal{L}\) is an ample line bundle, which restricts to a trivial line bundle on \(L/L \cap H\), and satisfies the numerical condition.

Version for pairs: characterization of log-KE metrics on such manifolds. Also coercivity of log-Mabuchi functional for different polarizations.
Horosymmetric varieties: examples I

Trivial fiber: generalized flag manifolds.
Trivial base: wonderful compactifications of symmetric spaces, but also

**Biequivariant group compactifications**

\[ X \curvearrowright G \times G \text{ with } x \in X \text{ such that } \overline{G \times G \cdot x} = X \text{ and } \text{Stab}_{G \times G}(x) = \text{diag}(G). \]

A version of the coercivity criterion is due to [Li, Zhou, Zhu] (there have subsequent on Mabuchi metrics [Li, Zhou] and Sasaki-Einstein metrics [Li, Zhu]).
Toric manifolds

[Wang Zhu 2004] A Fano toric manifold \((X, K_X^{-1})\) with associated moment polytope \(\Delta\) is Kähler-Einstein if and only if the barycenter of \(\Delta\) is the origin.

The condition is closed: indeed, [Zhou Zhu 2008] gives not an open condition but shows K-stable wrt special equivariant t.c. equivalent to coercivity on the range of l.b. considered!

In fact they have an open condition for coercivity of modified Mabuchi functional related to existence of extremal metrics (Weiyong He arXiv:1801.07636 for extremal version of Chen-Cheng).
Non-trivial fibration:

**Homogeneous toric bundles**

This is a $G$-manifold $X$ equipped with a $G$-equivariant map $X \to G/P$ to a generalized flag manifold, such that the action of $P$ on the fiber $Z$ factorizes by a torus $(\mathbb{C}^*)^r$ and the fiber is toric for this action.

**Horospherical varieties**

When the fibration structure above does not extend to the whole variety.
Remarkable Property

A $G$-stable hypersurface in a $G$-horosymmetric variety is also a $G$-horosymmetric variety.

Closure of orbits in wonderful compactifications

They are the model of (toroidal) horosymmetric varieties:
Given $I \subset \{1, \ldots, r\}$ there exists a parabolic subgroup $P_I$ of $G$, with $\sigma$-stable Levi subgroup $L_I$, such that the closure of the orbit corresponding to $I$ is an equivariant fibration over $G/P_I$ with fiber the wonderful compactification of $L'_I/(L'_I)^\sigma$.
($L'_I$ is the derived subgroup of $L_I$)
Thank you!

Reference: ArXiv:1712.00221