KÄHLER GEOMETRY OF HOROSYMMETRIC VARIETIES, AND APPLICATION TO MABUCHI'S K-ENERGY FUNCTIONAL

THIBAUT DELCROIX

Abstract. We introduce a class of almost homogeneous varieties contained in the class of spherical varieties and containing horospherical varieties as well as complete symmetric varieties. We develop Kähler geometry on these varieties, with applications to canonical metrics in mind, as a generalization of the Guillemin-Abreu-Donaldson geometry of toric varieties. Namely we associate convex functions with hermitian metrics on line bundles, and express the curvature form in terms of this function, as well as the corresponding Monge-Ampère volume form and scalar curvature. We then provide an expression for the Mabuchi functional and derive as an application a combinatorial sufficient condition of properness similar to one obtained by Li, Zhou and Zhu on group compactifications. This finally translates to a sufficient criterion of existence of constant scalar curvature Kähler metrics thanks to the recent work of Chen and Cheng. It yields infinitely many new examples of explicit Kähler classes admitting cscK metrics.

1. Introduction

Toric manifolds are complex manifolds equipped with an action of $(\mathbb{C}^*)^n$ such that there is a point with dense orbit and trivial stabilizer. The Kähler geometry of toric manifolds plays a fundamental role in Kähler geometry as a major source of examples as well as a testing ground for conjectures. It involves strong interactions with domains as various as convex analysis, real Monge-Ampère equations, combinatorics of polytopes, algebraic geometry of toric varieties, etc. The study of Kähler metrics on toric manifolds relies strongly on works of Guillemin, Abreu, then Donaldson. They have developped, using Legendre transform as a main tool, a very precise setting including:

- a model behavior for smooth Kähler metrics,
- a powerful expression of the scalar curvature,
- applications to the study of canonical Kähler metrics *via* the ubiquitous Mabuchi functional.

This setting allowed Donaldson to prove the Yau-Tian-Donaldson conjecture for constant scalar curvature Kähler (cscK) metrics on toric surfaces. That is, he showed that existence of cscK metrics in a given Kähler class corresponding to a polarization of a toric surface X by an ample line bundle L is equivalent to K-stability of (X, L) with respect to test configurations that are equivariant with respect to the torus action and further translated this condition into a condition on the associated polytope.

2010 *Mathematics Subject Classification.* 14M27; 32M12; 32Q15.

Key words and phrases. horosymmetric varieties; Mabuchi K-energy functional; spherical varieties; cscK metrics; log Kahler Einstein metrics.

Our goal in this article is to generalize this setting to a much larger class of varieties, that we introduce: the class of *horosymmetric varieties*.

Let G be a complex connected linear reductive group. A normal algebraic G-variety X is called spherical if any Borel subgroup B of G acts with an open dense orbit on X. Major subclasses of spherical varieties are given by biequivariant group compactifications, and horospherical varieties. A horospherical variety is a G-variety with an open dense orbit which is a G-homogeneous fibration over a generalized flag manifold with fiber a torus $(\mathbb{C}^*)^r$. The author's previous work on spherical varieties has highlighted how they provide a richer source of examples than toric varieties, with several examples of behavior which cannot appear for toric varieties. While it was possible to work on the full class of spherical manifolds, from the point of view of algebraic geometry, for the question of the existence of Kähler-Einstein metrics thanks to the proof of the Yau-Tian-Donaldson conjecture for Fano manifolds, it is necessary to develop Guillemin-Abreu-Donaldson theory to treat more general questions. It seems a very challenging problem to do this uniformly for all spherical varieties. On the other hand, the author did develop part of this setting for group compactifications and horospherical varieties.

Group compactifications do not share a nice property that toric, horospherical and spherical varieties possess and frequently used in Kähler geometry: a codimension one invariant irreducible subvariety in a group compactification leaves the class of group compactifications. We introduce the class of horosymmetric varieties as a natural subclass of spherical varieties containing horospherical varieties, group compactifications and more generally equivariant compactifications of symmetric spaces, which possesses the property of being closed under taking a codimension one invariant irreducible subvariety. The definition is modelled on the description of orbits of wonderful compactifications of adjoint (complex) symmetric spaces by De Concini and Procesi: they all have a dense orbit which is a homogeneous fibration over a generalized flag manifold, whose fibers are symmetric spaces. We say a normal G-variety is *horosymmetric* if it admits an open dense orbit which is a homogeneous fibration over a generalized flag manifold, whose fibers are symmetric spaces. Such a homogeneous space is sometimes called a parabolic induction from a symmetric space. Here we allow symmetric spaces under reductive groups, thus recovering horospherical varieties by considering $(\mathbb{C}^*)^r$ as a symmetric space for the group $(\mathbb{C}^*)^r$ and the involution $\sigma(t) = t^{-1}$. For the sake of giving precise statement in this introduction, let us introduce some notations. A horosymmetric homogeneous space is a homogeneous space G/H such that there exists

- a parabolic subgroup P of G, with unipotent radical P^u,
- a Levi subgroup L of P,
- and an involution of complex groups $\sigma : L \to L$,

such that $P^u \subset H$ and $(L^\sigma)^0 \subset L \cap H$ as a finite index subgroup, where L^σ denotes the subgroup of elements fixed by σ and $(L^\sigma)^0$ its neutral connected component.

Spherical varieties in general admit a combinatorial description: there is on one hand a complete combinatorial caracterization of spherical homogeneous spaces by Losev [Los09] and on the other hand a combinatorial classification of embeddings of a given spherical homogeneous space by Luna and Vust [LV83, Kno91]. General results about parabolic induction allow to derive easily the information about a horosymmetric homogeneous spaces from the information about the symmetric space fiber. For symmetric spaces, most of the information is contained in the restricted root system. Choose a torus $T_s \subset L$ on which the involution acts as the inverse, and maximal for this property. It is contained in a maximal σ-stable torus T in L. Then consider the restriction of roots of G (with respect to T) to T_s. Let Φ_s denote the subset of roots whose restriction are not identically zero. The restrictions of the roots in Φ_s form a (possibly non reduced) root system called the restricted root system, with corresponding notions of restricted Weyl group $\bar{W}$, restricted Weyl chambers, etc. Let $\mathfrak{Y}(T_s)$ denote the group of one-parameter subgroups of T_s, and identify $\mathfrak{a}_s = \mathfrak{Y}(T_s) \otimes \mathbb{R}$ with the Lie algebra of the non-compact part of the torus T_s. The image of $\mathfrak{a}_s$ by the exponential, then by the action on the base point $x \in X$, intersects every orbit of a maximal compact subgroup K of G along restricted Weyl group orbits (see Section 2 for details).

Let $\mathcal{L}$ be a G-linearized line bundle on a horosymmetric homogeneous space. It is determined by its isotropy character χ. To a K-invariant metric h on $\mathcal{L}$ we associate a function $u : \mathfrak{a}_s \to \mathbb{R}$, called the *toric potential*, which together with χ totally encodes the metric. One of our main result is the derivation of an expression of the curvature form ω of h in terms of its toric potential. We achieve this for the general case, but let us only give the statement in the nicer situation where the restriction of $\mathcal{L}$ to the symmetric space fiber is trivial. By fixing a choice of basis of a complement of $\mathfrak{h}$ in $\mathfrak{g}$, we may define reference real (1,1)-forms $\omega_{\heartsuit,\diamondsuit}$ indexed by couples of indices in $\{1,\dots,r\} \cup \Phi_s^+ \cup \Phi_{Q^u}$ (where $r = \dim(T_s)$, Φ_s^+ is the intersection of Φ_s with some system of positive roots, $\Phi_{Q^u} = -\Phi_{P^u}$ is the set of opposite of roots of P^u) that form a pointwise basis, and we express the curvature form in these terms. Given a root α of G, we denote by $\alpha^\vee$ its associated coroot.

Theorem 1.1. *Assume that the restriction of $\mathcal{L}$ to the symmetric fiber is trivial. Let $a \in \mathfrak{a}_s$ be such that $\beta(a) \neq 0$ for all $\beta \in \Phi_s$. Then*

$$\omega_{\exp(a)H} = \sum_{1\leq j_1,j_2\leq r} \frac{1}{4} d_a^2 u(l_{j_1}, l_{j_2}) \omega_{j_1,j_2} + \sum_{\alpha\in\Phi_{Q^u}} \frac{-e^{2\alpha}}{2} (d_a u - 2\chi)(\alpha^\vee) \omega_{\alpha,\alpha} + \sum_{\beta\in\Phi_s^+} \frac{d_a u(\beta^\vee)}{\sinh(2\beta(a))} \omega_{\beta,\beta}.$$

The previous theorem concerns only horosymmetric homogeneous spaces. To move on to horosymmetric varieties, we need more input from the general theory of spherical varieties. To a G-linearized line bundle on a horosymmetric variety are associated several polytopes. Of major importance is the (algebraic) moment polytope Δ^+, obtained as the closure of the set of suitably normalized highest weights of the spaces of multisections of $\mathcal{L}$, seen as G-representations. It lies in the real vector space $\mathfrak{X}(T) \otimes \mathbb{R}$, where $\mathfrak{X}(T)$ denotes the group of characters of T. The main application of the moment polytope to Kähler geometry is that it controls the asymptotic behavior of toric potentials, which in the case of positively curved metrics further allows to give a formula for integration with respect to the Monge-Ampère of the curvature from, in conjunction with the previous theorem. Again we do not state our results in their most general form in this introduction but prefer to give the general philosophy in a situation which is simpler than the general one.

Theorem 1.2. *Assume that $\mathcal{L}$ is an ample G-linearized line bundle on a non-singular horosymmetric variety X, and that it admits a global Q-semi-invariant holomorphic section, where Q is the parabolic opposite to P with respect to T. Let*

h be a smooth K-invariant metric on $\mathcal{L}$ with positive curvature ω and toric potential u. Then

(1) *u is smooth, $\bar{W}$-invariant and strictly convex,*
(2) *$a \mapsto d_a u$ defines a diffeomorphism from $\mathrm{Int}(-\mathfrak{a}_s^+)$ onto $\mathrm{Int}(2\chi - 2\Delta^+)$.*

Let ψ denote a K-invariant function on X, integrable with respect to ω^n. Let dq denote the Lebesgue measure on the affine span of Δ^+, normalized by the lattice $\chi + \mathfrak{X}(T/T \cap H)$. Then there exist a constant C, independent of h and ψ, such that

$$\int_X \psi \omega^n = C \int_{\Delta^+} \psi(d_{2\chi - 2q} u^*) P_{DH}(q) dq.$$

where $P_{DH}(q) = \prod_{\alpha \in \Phi_{Q^u} \cup \Phi_s^+} \kappa(\alpha, q)/\kappa(\alpha, \rho)$, ρ is the half sum of positive roots of G and u^* is the convex conjugate of u.

The two theorems above form a strong basis to attack Kähler geometry questions on horosymmetric varieties. They are for example all that is needed to study the existence of Fano Kähler-Einstein metrics with the strategy following the lines of Wang and Zhu's work on toric Fano manifolds. They also allow to push further, namely to compute an expression of the scalar curvature, to compute an expression of the (log)-Mabuchi functional, then to obtain a coercivity criterion for this functional, in the line of work of Li-Zhou-Zhu for group compactifications.

The Mabuchi functional is a functional on the space of Kähler metrics in a given class, whose smooth minimizers if they exist should be cscK metrics. There are several extensions of this notion, in particular the log-Mabuchi functional, related to the existence of log-Kähler-Einstein metrics. A natural way to search for minimizers of this functional is to try to prove its properness, or coercivity, with respect to the J-functional. The J-functional is another standard functional in Kähler geometry, which may be considered as a measure of distance from a fixed reference metric in the space of Kähler metrics.

We provide in this paper an application of our setting to this problem of coercivity of the Mabuchi functional, obtaining a very general, but at the same time far from optimal, coercivity criterion for the Mabuchi functional on horosymmetric varieties. We work under several simplifying assumptions to carry out the proof while keeping a reasonnable length for the paper, but expect that several of these assumptions can be removed with a little work.

Instead of stating these assumptions in this introduction, let us state the result in three examples of situations where they are satisfied. They are as follows, in all cases, G is a complex connected linear reductive group and X is a smooth projective G-variety.

(1) The manifold X is a group compactification, that is, $G = G_0 \times G_0$ and there exists a point $x \in X$ with stabilizer $\mathrm{diag}(G_0) \subset G$ and dense orbit. We may consider any ample G-linearized line bundle on X.
(2) The manifold X is a homogeneous toric bundle under the action of G, that is there exists a projective homogeneous G/P and a G-equivariant surjective morphism $X \to G/P$ with fiber isomorphic to a toric variety, under the action of P which factorizes through a torus $(\mathbb{C}^*)^r$. We consider any ample G-linearized line bundles on X.
(3) The manifold X is a toroidal symmetric variety of type $\mathrm{AIII}(r, m > 2r)$, that is, m and r are two positive integers with $m > 2r$, $G = \mathrm{SL}_m$, there exists a point $x \in X$ with dense orbit, whose orbit is isomorphic

to $\mathrm{SL}_m/\mathrm{S}(\mathrm{GL}_r\times\mathrm{GL}_{m-r})$, and there exists a dominant G-equivariant morphism from X to the wonderful compactification of this symmetric space. We may consider any ample G-linearized line bundle which restricts to a trivial line bundle on the dense orbit.

Let Θ be a G-equivariant boundary, that is, an *effective* $\mathbb{Q}$-divisor $\Theta=\sum_Y c_y Y$ where Y runs over all G-stable irreducible codimension one submanifolds of X. We assume furthermore that the support of Θ is simple normal crossing and $c_Y<1$ for all Y. In particular, the pair (X,Θ) is klt. It follows from the combinatorial description of horosymmetric varieties that to each Y as above is associated an element μ_Y of $\mathfrak{Y}(T_s)$. Let Δ^+ be the moment polytope of $\mathcal{L}$, and let λ_0 be a well chosen point in Δ^+ (see Section 7). Let $\tilde{\Delta}^+_Y$ denote the bounded cone with vertex λ_0 and base the face of Δ^+ whose outer normal is $-\mu_Y$ in the affine space $\chi+\mathfrak{X}(T/T\cap H)\otimes\mathbb{R}$. Let χ^{ac} denote the restriction of the character $\sum_{\alpha\in\Phi_{Q^u}}\alpha$ of P to H, and set

$$\Lambda_Y=\frac{-c_Y+1-\sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha(\mu_Y)}{\sup\{p(\mu_Y),p\in\chi-\Delta^+\}},$$

$$I_H(a)=\sum_{\beta\in\Phi_s^+}\ln\sinh(-2\beta(a))-\sum_{\alpha\in\Phi_{Q^u}}2\alpha(a),$$

and

$$\bar{S}_\Theta=\bar{S}-n\sum_Y c_Y\mathcal{L}|_Y^{n-1}/\mathcal{L}^n.$$

Let Mab_Θ denote the log-Mabuchi functional in this setting, and write $\mathrm{Mab}_\Theta(u)$ for its value on the hermitian metric with toric potential u. For the question of coercivity, the log-Mabuchi functional matters only up to normalizing additive and multiplicative constants, so we ignore these in the statement.

Theorem 1.3. *Let $p=p(q):=2(\chi-q)$, then we have*

$$\begin{aligned}\mathrm{Mab}_\Theta(u)=&\sum_Y\Lambda_Y\int_{\tilde{\Delta}^+_Y}(nu^*(p)-u^*(p)\sum\frac{\chi(\alpha^\vee)}{q(\alpha^\vee)}+d_pu^*(p))P_{DH}(q)dq\\&+\int_{\Delta^+}u^*(p)(\sum\frac{\chi^{ac}(\alpha^\vee)}{q(\alpha^\vee)}-\bar{S}_\Theta)P_{DH}(q)dq-\int_{\Delta^+}I_H(d_pu^*)P_{DH}(q)dq\\&-\int_{\Delta^+}\ln\det(d_p^2u^*)P_{DH}(q)dq\end{aligned}$$

where u^* denotes the Legendre transform, or convex conjugate, of u.

As an application of this formula, we obtain the following sufficient condition for coercivity. Consider the function $F_\mathcal{L}$ defined piecewise by

$$F_\mathcal{L}(q)=(n+1)\Lambda_Y-\bar{S}_\Theta+\sum\frac{(\chi^{ac}-\Lambda_Y\chi)(\alpha^\vee)}{q(\alpha^\vee)}$$

for q in $\tilde{\Delta}^+_Y$. Define an element bar of the affine space generated by Δ^+ by setting

$$\mathrm{bar}=\int_{\Delta^+}qF_\mathcal{L}(q)\frac{P_{DH}(q)dq}{\int_{\Delta^+}P_{DH}dq}.$$

Despite the notation, it is not in general the barycenter of Δ^+ with respect to a positive measure. We will however see how to consider it as a barycenter in the article. The coercivity criterion is stated in terms of $F_{\mathcal{L}}$ and bar. Let $2\rho_H$ denote the element of $\mathfrak{a}_s^*$ defined by the restriction of $\sum_{\alpha\in\Phi_{Q^u}\cup\Phi^+}\alpha$ to $\mathfrak{a}_s$.

Theorem 1.4. *Assume that $F_{\mathcal{L}}>0$ and that the point*

$$(\min_Y \Lambda_Y)\frac{\int_{\Delta^+} P_{DH}dq}{\int_{\Delta^+} F_{\mathcal{L}}P_{DH}dq}(\mathrm{bar}-\chi)-2\rho_H$$

is in the relative interior of the dual cone of $\mathfrak{a}_s^+$. Then the Mabuchi functional is proper modulo the action of $Z(L)^0$, and there exist constant scalar curvature Kähler metrics in $c_1(\mathcal{L})$.

The last conclusion follows from the first and the recent breakthrough of Chen and Cheng [CC18]. This allows to obtain infinite families of new examples of classes with constant scalar curvature metrics.

In the case when $\mathcal{L}=K_X^{-1}\otimes\mathcal{O}(-\Theta)$ is ample, the pair (X,Θ) is log-Fano and the minimizer of the log-Mabuchi functional are log-Kähler-Einstein metrics. In this situation it is known that properness of the log-Mabuchi functional implies the existence of log-Kähler-Einstein metrics (see [Dar17] for a statement allowing automorphisms), so we obtain the following corollary.

Theorem 1.5. *Assume $\mathcal{L}=K_X^{-1}\otimes\mathcal{O}(-\Theta)$, then (X,Θ) admits a log-Kähler-Einstein metric provided* bar $-\sum_{\alpha\in\Phi_{Q^u}\cup\Phi^+}\alpha$ *is in the relative interior of the dual cone of $\mathfrak{a}_s^+$.*

If one works on, say, a biequivariant compactification of a semisimple group, then it is not hard to check that the condition above is open as $\mathcal{L}$ and Θ vary. Starting from an example of Kähler-Einstein Fano manifold obtained in [Del17], we can extract from this corollary an explicit subset of K_X^{-1} in the ample cone, with non-empty interior, such that each corresponding $\mathcal{L}$ writes as $\mathcal{L}=K_{(X,\Theta)}^{-1}$ and the pair (X,Θ) admits a log-Kähler-Einstein metric.

While the point of view adopted in this article is definitely in line with the author's earlier work on group compactifications and horospherical varieties, it should be mentionned that there were previous works, and different perspectives on both these classes. Group compactifications have been studied in detail from the algebraic point of view and the first article about the existence of canonical metrics on these was [AK05] to the author's knowledge, and it builded on the extensive study of reductive varieties in [AB04a, AB04b]. Homogeneous toric bundles have been studied for the Kähler-Einstein metric existence problem by Podesta and Spiro [PS10], their point of view on the Kähler geometry of this subclass of horospherical varieties being somewhat different from the author's. Donaldson highlighted in [Don08] the importance and studying these varieties, and there were partly unpublished work of Raza and Nyberg on these subjects in their PhD theses [Raz, Raz07, Nyb]. Finally, concerning the application to the Mabuchi functional, we were strongly influenced by Li, Zhou and Zhu's article [LZZ]. The latter in turn used as foundations on one side our work on group compactifications and on the other side a strategy for obtaining coercivity of the Mabuchi functional developped initially by Zhou and Zhu [ZZ08]. It should be noted that the criterion we obtain for (non-semi-simple) group compactifications is *a priori* not equivalent to the one given in [LZZ]. We do not claim that ours is better but only that theirs did not generalize naturally to our broader setting.

The paper is organized as follows. Section 2 is devoted to the introduction of horosymmetric homogeneous spaces, and of the combinatorial data associated to them. In Section 3, we introduce the toric potential of a K-invariant metric on a G-linearized line bundle on a horosymmetric homogeneous space, and compute the curvature form of such a metric in terms of this function. Even though the proof is rather technical, involving a lot of Lie bracket computations, it is a central part of the theory to have this precise expression. Theorem 1.1 is Corollary 3.11, a special case of Theorem 3.10. In Section 4, we switch to horosymmetric varieties, we recall their combinatorial classification inherited from the theory of spherical varieties, and we check that a G-invariant irreducible codimension one subvariety remains horosymmetric. Section 5 presents the combinatorial data associated with line bundles on horosymmetric varieties, and in particular the link between several convex polytopes associated to such a line bundle. Section 6 applies the previous sections to hermitian metrics on polarized horosymmetric varieties, to obtain the behavior of toric potentials and an integration formula. In particular, Theorem 1.2 is proved here (Proposition 6.3 and Proposition 6.9). Finally, we give in Section 7 the application to the Mabuchi functional, starting with a computation of the scalar curvature, then of the Mabuchi functional, to arrive to a coercivity criterion. Theorem 1.3, Theorem 1.4 and Theorem 1.5 are proved in this final section (respectively in Theorem 7.4, Theorem 7.9 and Corollary 7.13).

We tried to illustrate all notions by simple examples (even if they sometimes appear trivial, we believe they are essential to make the link between the theory of spherical varieties and standard examples of complex geometry) and to follow for the whole paper the example of symmetric varieties of type AIII.

2. Horosymmetric homogeneous spaces

In this section we introduce horosymmetric homogeneous spaces and their associated combinatorial data, extracting from the literature the results needed for the next sections.

2.1. **Definition and examples.** We always work over the field $\mathbb{C}$ of complex numbers. Given an algebraic group G, we denote by G^u its *unipotent radical*. A subgroup L of G is called a *Levi subgroup* of G if it is a reductive subgroup of G such that G is isomorphic to the semidirect product of G^u and L. There always exists a Levi subgroup, and any two Levi subgroups are conjugate by an element of G^u. A complex algebraic group G is called *reductive* if G^u is trivial.

From now on and for the whole paper, G will denote a connected, reductive, complex, linear algebraic group. Recall that a *parabolic subgroup* of G is a closed subgroup P such that the corresponding homogeneous space G/P is a projective manifold, called a *generalized flag manifold*. Recall also for later use that a *Borel subgroup* of G is a parabolic subgroup which is minimal with respect to inclusion. Note that any parabolic subgroup of G contains at least one Borel subgroup of G.

Definition 2.1. A closed subgroup H of G is a *horosymmetric subgroup* if there exists a parabolic subgroup P of G, a Levi subgroup L of P and a complex algebraic group involution σ of L such that

- $P^u \subset H \subset P$ and
- $(L^\sigma)^0 \subset L \cap H$ as a finite index subgroup,

where L^σ denotes the subgroup of elements fixed by σ and $(L^\sigma)^0$ its neutral connected component.

Remark 2.2. The condition of being horosymmetric may be read off directly from the Lie algebra of H. As a convention, we denote the Lie algebra of a group by the same letter, in fraktur gothic lower case letter. Then H is symmetric if and only if there exists a parabolic subgroup P, a Levi subgroup L of P, and a complex Lie algebra involution σ of $\mathfrak{l}$ such that

$$\mathfrak{h} = \mathfrak{p}^u \oplus \mathfrak{l}^\sigma$$

From now on, H will denote a horosymmetric subgroup, and P, L, σ will be as in the above definition. We keep the same notation σ for the induced involution of the Lie algebra $\mathfrak{l}$. We will also say that G/H is a *horosymmetric homogeneous space*.

Note that $L \cap H \subset N_L(L^\sigma)$, and we have the following description of $N_L(L^\sigma)$, due to De Concini and Procesi. They assume in their paper that G is semisimple but the proof applies to reductive groups just as well.

Proposition 2.3 ([DP83])**.** *The normalizer $N_L(L^\sigma)$ is equal to the subgroup of all g such that $g\sigma(g)^{-1}$ is in the center of L.*

In particular if $L = G$ is semisimple, then $N_L(L^\sigma)/(L^\sigma)^0$ is finite. Note also that if in addition L is adjoint, $N_L(L^\sigma) = L^\sigma$ and if L is simply connected, then L^σ is connected.

Example 2.4. Trivial examples of horosymmetric subgroups are obtained by setting $\sigma = \mathrm{id}_L$. Then $H = P$ is a parabolic subgroup and G/H is a generalized flag manifold. Since we will use them later, let us recall a fundamental example of flag manifold: the Grassmannian $\mathrm{Gr}_{r,m}$ of r-dimensional linear subspaces in $\mathbb{C}^m$, under the action of SL_m. The stabilizer of a point is a proper, maximal (with respect to inclusion) parabolic subgroup of SL_m (for $1 \leq r \leq m-1$).

Example 2.5. Assume that $G = (\mathbb{C}^*)^n$, then $P = L = G$. If we consider the involution defined by $\sigma(g) = g^{-1}$, which is an honest complex algebraic group involution since G is abelian, we obtain $\{e\} \subset H \subset \{\pm 1\}^n$ and in any case $G/H \simeq (\mathbb{C}^*)^n$. Hence a torus may be considered as a horosymmetric homogeneous space.

Let $[L, L]$ denote the derived subgroup of L and $Z(L)$ the center of L. Then L is a semidirect product of these two subgroups, which means, at the level of Lie algebras, that

$$\mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{z}(\mathfrak{l}).$$

Note that any involution of L preserves this decomposition.

Example 2.6. A closed subgroup of G is called *horospherical* if it contains the unipotent radical of a Borel subgroup of G.

Assume that the involution σ of L restricts to the identity on $[\mathfrak{l}, \mathfrak{l}]$. Then H contains the unipotent radical of any Borel subgroup contained in P. Hence H is horospherical.

Conversely, if H is a horospherical subgroup of G, then taking $P := N_G(H)$ which is a parabolic subgroup of G, and letting L be any Levi subgroup of P, we have $\mathfrak{h} = \mathfrak{p}^u \oplus [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{b}$ where $\mathfrak{b} = \mathfrak{h} \cap \mathfrak{z}(\mathfrak{l})$ (see [Pas08, Section 2]). Choose any complement $\mathfrak{c}$ of $\mathfrak{b}$ in $\mathfrak{z}(\mathfrak{l})$, and consider the involution of $\mathfrak{l}$ defined as id on $[\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{b}$ and as $-\mathrm{id}$ on $\mathfrak{c}$. This shows that H is a horosymmetric subgroup of G.

Example 2.7. Consider the linear action of SL_2 on $\mathbb{C}^2 \setminus \{0\}$. It is a transitive action and the stabilizer of $(1,0)$ is the unipotent subgroup B^u of the Borel subgroup formed by upper triangular matrices. Under this action, $\mathbb{C}^2 \setminus \{0\}$ is a horospherical, hence horosymmetric, homogeneous space. Alternatively, one may consider the action of GL_2 instead of the action of SL_2.

Example 2.8. Assume $P = L = G$, then σ is an involution of G, and $(G^\sigma)^0 \subset H \subset N_G((G^\sigma)^0)$. These subgroups are commonly known as *symmetric subgroups* and the associated homogeneous spaces as (complex reductive) *symmetric spaces.*

All horosymmetric homogeneous spaces may actually be considered as parabolic inductions from symmetric spaces. Let us recall the definition of parabolic induction.

Definition 2.9. Let G and L be two reductive algebraic groups, then we say that a G-variety X is obtained from an L-variety Y by *parabolic induction* if there exists a parabolic subgroup P of G, and an surjective group morphism $P \to L$ such that $X = G *_P Y$ is the G-homogeneous fiber bundle over G/P with fiber Y.

In our situation, G/H admits a natural structure of G-homogeneous fiber bundle over G/P, with fiber P/H. The action of P on P/H factorizes by P/P^u and under the natural isomorphism $L \simeq P/P^u$, identifies the fiber with the L-variety $L/L\cap H$, which is a symmetric homogeneous space. Conversely, any parabolic induction from a symmetric space is a horosymmetric homogeneous space. The special case of horospherical homogeneous spaces consists of parabolic inductions from tori.

We will denote by f the G-equivariant map $G/H \to G/P$ and by π the quotient map $G \to G/H$.

Let us now give more explicit examples of horosymmetric homogeneous spaces, starting by examples of symmetric spaces.

Example 2.10. Assume $\mathfrak{g} = \mathfrak{sl}_m$ for some m. Then there are three families of group involutions of $\mathfrak{g}$ up to conjugation [GW09, Sections 11.3.4 and 11.3.5]. For a nicer presentation we work on the group $G = \mathrm{SL}_m$. For an integer $p > 0$, we define the $2p \times 2p$ block diagonal matrix T_p by $T_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $T_p = \mathrm{diag}(T_1, \dots, T_1)$. For an integer $0 < r < m/2$, we define the $m \times m$ matrix J_r as follows. Let S_r be the $r \times r$ matrix with coefficients $(\delta_{j+k,r+1})_{j,k}$, and set

$$J_r = \begin{pmatrix} 0 & 0 & S_r \\ 0 & I_{n-2r} & 0 \\ S_r & 0 & 0 \end{pmatrix}.$$

The types of involutions are the following:

(1) (Type AI(m)) Consider the involution of G defined by $\sigma(g) = (g^t)^{-1}$ where $\cdot^t$ denotes the transposition of matrices. Then $G^\sigma = \mathrm{SO}_m$. The symmetric space $G/N_G(G^\sigma)$ may be identified with the space of non-degenerate quadrics in $\mathbb{P}^{m-1}$, equipped with the action of G induced by its natural action on $\mathbb{P}^{m-1}$.
(2) (Type AII(p)) Assume $n = 2p$ is even. Let σ be the involution defined by $\sigma(g) = T_p(g^t)^{-1}T_p^t$. Then $G^\sigma = \mathrm{Sp}_{2p}$ is the group of elements that preserve the non-degenerate skew-symmetric bilinear form $\omega(u,v) = u^t T_p v$ on $\mathbb{C}^{2p}$.
(3) (Type AIII(r,m)) Let σ be the involution $g \mapsto J_r g J_r$. Then G^σ is conjugate to the subgroup $S(\mathrm{GL}_r \times \mathrm{GL}_{m-r})$.

The space G/G^σ may be considered as the set of pairs (V_1, V_2) of linear subspaces $V_j \subset \mathbb{C}^m$ of dimension $\dim(V_1) = r$, $\dim(V_2) = m - r$, such that $V_1 \cap V_2 = \{0\}$. This is an (open dense) orbit for the diagonal action of G on the product of Grassmannians $\mathrm{Gr}_{r,m} \times \mathrm{Gr}_{m-r,m}$.

Example 2.11. Let us illustrate the characterization of the normalizer of a symmetric subgroup in type AIII case. First, since $G = \mathrm{SL}_n$ is simply connected, G^σ is connected. Furthermore, it is easy to check here that $N_G(G^\sigma)$ is different from G^σ if and only if n is even and $r = n/2$, in which case G^σ is of index two in $N_G(G^\sigma)$. For example, if $n = 2$ and $r = 1$, $N_G(G^\sigma)$ is generated by G^σ and $\mathrm{diag}(i, -i)$. In that situation, $G/N_G(G^\sigma)$ is the space of unordered pairs $\{V_1, V_2\}$ of linear subspaces $V_j \subset \mathbb{C}^m$ of dimension r for $j = 1$ and $m - r$ for $j = 2$, such that $V_1 \cap V_2 = \{0\}$.

Example 2.12. Finally, let us give an explicit example of non trivial parabolic induction from a symmetric space. Consider the subgroup H of SL_3 defined as the set of matrices of the form

$$\begin{pmatrix} a & b & e \\ b & a & f \\ 0 & 0 & g \end{pmatrix}.$$

Then obviously H is contained in the parabolic P composed of matrices with zeroes where the general matrix of H has zeroes, and contains its unipotent radical, which consists of the matrices as above with $a = g = 1$ and $b = 0$. The subgroup $L = \mathrm{S}(\mathrm{GL}_2 \times \mathbb{C}^*)$ is then a Levi subgroup of P and $L \cap H$ is the subgroup of elements of L fixed by the involution $g \mapsto MgM$ where

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.2. **Root systems.**

2.2.1. *Maximally split torus.* A torus T in L is *split* if $\sigma(t) = t^{-1}$ for any $t \in T$. A torus T in L is *maximally split* if T is a σ-stable maximal torus in L which contains a split torus T_s of maximal dimension among split tori. It turns out that any split torus is contained in a σ-stable maximal torus of L [Vus74] hence maximally split tori exist. From now on, T denotes a maximally split torus in L with respect to σ, and T_s denotes its maximal split subtorus. If T^σ denotes the subtorus of elements of T fixed by σ, then $T^\sigma \times T_s \to T$ is a surjective morphism, with kernel a finite subgroup. The dimension of T_s is called the *rank* of the symmetric space $L/L \cap H$.

Example 2.13. The ranks and maximal tori for involutions of SL_n are as follows.

- (Type AI(m)) For $\sigma : g \mapsto (g^t)^{-1}$, the rank is $m - 1$ and the torus T of diagonal matrices is a split torus which is also maximal, hence $T_s = T$ in this case.
- (Type AII(p)) For $\sigma : g \mapsto T_p(g^t)^{-1}T_p^t$, with $m = 2p$, the rank is $p - 1$, and the torus of diagonal matrices provides a maximally split torus. The maximal split subtorus T_s is then the subtorus of diagonal matrices of the form
$$\mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_p, a_p)$$
with $a_1, \dots, a_{p-1} \in \mathbb{C}^*$ and $a_p = (a_1^2 \cdots a_{p-1}^2)^{-1}$, and T^σ is the subtorus of diagonal matrices of the form $\mathrm{diag}(a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_p, a_p^{-1})$ with $a_1, \dots, a_p \in \mathbb{C}^*$. We record for later use that $\sigma(\mathrm{diag}(a_1, \dots, a_n)) = \mathrm{diag}(a_2^{-1}, a_1^{-1}, a_4^{-1}, \dots, a_{n-1}^{-1})$.

- (Type AIII(r, m) Finally, for $\sigma : g \mapsto J_r g J_r$, the rank is r, and the torus T of diagonal matrices is again maximally split. Let υ denote the permutation of $\{1, \dots, m\}$ defined by $\upsilon(i) = m+1-i$ if $1 \leq i \leq r$ or $m+1-r \leq i \leq m$, and $\upsilon(i) = i$ otherwise. Then σ acts on diagonal matrices as
$$\sigma(\mathrm{diag}(a_1, \dots, a_m)) = \mathrm{diag}(a_{\upsilon(1)}, \dots, a_{\upsilon(m)}).$$
We then see that the subtorus T^σ is the torus of diagonal matrices of the form $\mathrm{diag}(a_1, a_2, \dots, a_{m-r}, a_r, a_{r-1}, \dots, a_1)$ and that T_s is the subtorus of diagonal matrices of the form $\mathrm{diag}(a_1, \dots, a_r, 1, \dots, 1, a_r^{-1}, \dots, a_1^{-1})$.

2.2.2. *Root systems and Lie algebras decompositions.* We denote by $\mathfrak{X}(T)$ the group of characters of T, that is, algebraic group morphisms from T to $\mathbb{C}^*$. We denote by $\Phi \subset \mathfrak{X}(T)$ the root system of (G, T). Recall the root space decomposition of $\mathfrak{g}$:
$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \{x \in \mathfrak{g}; \quad \mathrm{Ad}(t)(x) = \alpha(t)x \quad \forall t \in T\}$$
where Ad denotes the adjoint representation of G on $\mathfrak{g}$.

Example 2.14. In our examples we concentrate on the case when $G = \mathrm{SL}_n$, and the root system is of type A_{n-1}. Let us recall its root system with respect to the maximal torus of diagonal matrices, in order to fix the notations to be used in examples throughout the article. The roots are the group morphisms $\alpha_{j,k} : T \to \mathbb{C}^*$, for $1 \leq j \neq k \leq n$, defined by $\alpha_{j,k}(\mathrm{diag}(a_1, \dots, a_n)) = a_j / a_k$. The root space $\mathfrak{g}_{\alpha_{j,k}}$ is then the set of matrices with only one non zero coefficient at the intersection of the j^{th}-line and k^{th}-column.

We denote by $\Phi_L \subset \Phi$ the root system of L with respect to T, by $\Phi_{P^u} \subset \Phi$ the set of roots of P^u, so that
$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_L} \mathfrak{g}_\alpha, \qquad \mathfrak{p} = \mathfrak{l} \oplus \bigoplus_{\alpha \in \Phi_{P^u}} \mathfrak{g}_\alpha$$
and
$$\mathfrak{h} = \mathfrak{l} \cap \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{P^u}} \mathfrak{g}_\alpha.$$

Example 2.15. In the case of Example 2.12, $G = \mathrm{SL}_3$ and T is the torus of diagonal matrices. Using notations from Example 2.14, we have $\Phi = \{\pm\alpha_{1,2}, \pm\alpha_{2,3}, \pm\alpha_{1,3}\}$, $\Phi_L = \{\pm\alpha_{1,2}\}$, $\Phi_{P^u} = \{\alpha_{1,3}, \alpha_{2,3}\}$.

2.2.3. *Restricted root system.* The set of roots in Φ_L fixed by σ is a sub root system denoted by Φ_L^σ. Let $\Phi_s = \Phi_L \setminus \Phi_L^\sigma$. Note that Φ_s is not a root system in general. Let us now introduce the restricted root system of $L/L \cap H$. Given $\alpha \in \Phi_L$, we set $\bar{\alpha} = \alpha - \sigma(\alpha)$. It is zero if and only if $\alpha \in \Phi_L^\sigma$.

Proposition 2.16 ([Ric82, Section 4])**.** *The set*
$$\bar{\Phi} = \{\bar{\alpha}; \alpha \in \Phi_s\} \subset \mathfrak{X}(T)$$
is a (possibly non reduced) root system in the linear subspace of $\mathfrak{X}(T) \otimes \mathbb{R}$ it generates. The Weyl group $\bar{W}$ of the root system $\bar{\Phi}$ may be identified with $N_L(T_s)/Z_L(T_s)$ and furthermore any element of $\bar{W}$ admits a representant in $N_{(L^\sigma)^0}(T_s)$.

The root system $\bar{\Phi}$ is called the *restricted root system* of the symmetric space $L/L\cap H$. We will also say that its elements are *restricted roots*, that $\bar{W}$ is the *restricted Weyl group*, etc.

Another interpretation of the restricted root system, which justifies the name, is obtained as follows. For any $t\in T_s$, and $\alpha\in\Phi_L$, we have

$$\sigma(\alpha)(t)=\alpha(\sigma(t))=\alpha(t^{-1})=(-\alpha)(t).$$

As a consequence, $\bar{\alpha}|_{T_s}=2\alpha|_{T_s}$, that is, up to a factor two, $\bar{\alpha}$ encodes the restriction of α to T_s. More significantly, given $\gamma\in\mathfrak{X}(T_s)$, let $\bar{\mathfrak{l}}_\gamma$ denote the subset of elements x in $\mathfrak{l}$ such that $\mathrm{Ad}(t)(x)=\gamma(t)x$ for all $t\in T_s$. Then by simultaneous diagonalization, we check that $\mathfrak{l}=\bigoplus_{\gamma\in\mathfrak{X}(T_s)}\bar{\mathfrak{l}}_\gamma$. We immediately remark that $\bar{\mathfrak{l}}_0$ contains $\mathfrak{t}$ and all $\mathfrak{g}_\alpha$ for $\alpha\in\Phi_L^\sigma$, and that $\bar{\mathfrak{l}}_\gamma$ contains $\mathfrak{g}_\alpha$ as soon as $\alpha\in\Phi_L$ is such that $\bar{\alpha}|_{T_s}=2\gamma$. By the usual root decomposition of $\mathfrak{l}$, we check that actually

$$\bar{\mathfrak{l}}_0=\mathfrak{t}\oplus\bigoplus_{\alpha\in\Phi_L^\sigma}\mathfrak{g}_\alpha,\qquad \bar{\mathfrak{l}}_\gamma=\bigoplus_{\bar{\alpha}|_{T_s}=2\gamma}\mathfrak{g}_\alpha$$

for $\gamma\neq 0$, and

$$\mathfrak{l}=\bar{\mathfrak{l}}_0\oplus\bigoplus_{\bar{\alpha}\in\bar{\Phi}}\bar{\mathfrak{l}}_{\bar{\alpha}/2}.$$

A restricted root $\bar{\alpha}$ is fully determined by its restriction to T_s since $T=T_sT^\sigma$ and $\bar{\alpha}|_{T^\sigma}=0$ since $\sigma(\bar{\alpha})=-\bar{\alpha}$.

Example 2.17. In the case of type AI(m), Φ_L^σ is empty and $\Phi_s=\Phi_L=\Phi$. For any $\alpha\in\Phi$, we have $\sigma(\alpha)=-\alpha$, hence the restricted root system is just the double 2Φ of Φ.

Example 2.18. In the case of type AII(p), we check that

$$\sigma(\alpha_{j,k})=\alpha_{k+(-1)^{k+1},j+(-1)^{j+1}}.$$

In fact it is easier to identify the restricted root system by analysing the restriction of roots to T_s. We denote an element of T_s by $\mathrm{diag}(b_1,b_1,b_2,b_2,\dots,b_p,b_p)$. We check easily that, for $1\leq j\neq k\leq p$,

$$\alpha_{2j,2k-1}|_{T_s}=\alpha_{2j-1,2k-1}|_{T_s}=\alpha_{2j-1,2k}|_{T_s}=\alpha_{2j,2k}|_{T_s}=b_j/b_k.$$

We deduce that $\Phi_L^\sigma=\{\pm\alpha_{2j-1,2j};1\leq j\leq p\}$ and that the restricted root system is of type A_{p-1}, with elements $\bar{\alpha}_{2j,2k}:\mathrm{diag}(b_1,b_1,b_2,b_2,\dots,b_p,b_p)\mapsto b_j^2/b_k^2$ for $1\leq j\neq k\leq p$.

Example 2.19. In the case of type AIII(r, m), finally, we will also identify the root system *via* restriction to T_s. We will denote an element of T_s, which is a diagonal matrix, by $\mathrm{diag}(b_1,\dots,b_r,1\dots,1,b_r^{-1},\dots,b_1^{-1})$. In the case when $m=2r$, there are no 1 in the middle and the restricted root system will be slightly different.

In general, the restriction $\alpha_{j,k}|_{T_s}$ is trivial if and only if $r+1\leq j\neq k\leq m-r$, which proves Φ_L^σ is the subsystem formed by these roots.

Since $\alpha_{j,k}=-\alpha_{k,j}$, it is obviously enough to consider only the case when $j<k$. For $1\leq j<k\leq r$, we have $\alpha_{j,k}|_{T_s}=\alpha_{m-k+1,m-j+1}|_{T_s}=b_j/b_k$. For $1\leq j\leq r$ and $r+1\leq k\leq m-r$, we have $\alpha_{j,k}|_{T_s}=\alpha_{m-k+1,m-j+1}|_{T_s}=b_j$. Finally, for $1\leq j\leq r$ and $m+1-r\leq k\leq m$, we have $\alpha_{j,k}|_{T_s}=\alpha_{m-k+1,m-j+1}|_{T_s}=b_jb_{m+1-k}$. Remark that in this last case, we may have $\alpha_{m-k+1,m-j+1}=\alpha_{j,k}$, namely when $j=m+1-k$. In this situation we obtain the function b_j^2. Hence, whenever $r+1\leq m-r$, or equivalently $r<m/2$ since both r and m are integers, the

restricted root system is non reduced. It is possible to check that it is of type BC_r. In the remaining case, that is when $n = 2r$, the restricted root system is of type C_r.

2.3. **Cartan involution and fundamental domain.** There always exists a Cartan involution of G such that its restriction to L commutes with σ. We fix such a Cartan involution θ. Denote by $K = G^\theta$ the corresponding maximal compact subgroup of G. Let $\mathfrak{a}_s$ denote the Lie subalgebra $\mathfrak{t}_s \cap i\mathfrak{k}$ of $\mathfrak{t}_s$.

Consider the group $\mathfrak{Y}(T_s)$ of one-parameter subgroups of T_s, that is, algebraic group morphisms $\mathbb{C}^* \to T_s$. This group naturally embeds in $\mathfrak{a}_s$: given $\lambda \in \mathfrak{Y}(T_s)$, it induces a Lie algebra morphism $d_e\lambda : \mathbb{C} \to \mathfrak{t}_s$. Here we identified the Lie algebra of $\mathbb{C}^*$ with $\mathbb{C}$ and the exponential map is given by the usual exponential. Then $d_e\lambda(1)$ must be an element of $\mathfrak{a}_s$ and it determines λ completely. This induces an injection of $\mathfrak{Y}(T_s)$ in $\mathfrak{a}_s$ which actually allows to identify $\mathfrak{a}_s$ with $\mathfrak{Y}(T_s) \otimes \mathbb{R}$.

Recall that we may either consider the restricted root system $\bar{\Phi}$ as in $\mathfrak{X}(T)$, in which case it lies in the subgroup $\mathfrak{X}(T/T \cap H)$, or, *via* the restriciton to T_s, we may consider $\bar{\Phi}$ to be in $\mathfrak{X}(T_s)$. This allows to define a Weyl chambers in $\mathfrak{a}_s$ with respect to the restricted root system. Choose any such Weyl chamber, denote it by $\mathfrak{a}_s^+$ and call it the *positive restricted Weyl chamber*.

Proposition 2.20. *The natural map $\mathfrak{a}_\mathfrak{s} \to \exp(\mathfrak{a}_s)H/H$ is injective, and the intersection of a K-orbit in G/H with $\exp(\mathfrak{a}_s)H$ is the image by this map of a $\bar{W}$-orbit in $\mathfrak{a}_s$. As a consequence, the subset $\exp(\mathfrak{a}_s^+)H/H$ is a fundamental domain for the action of K on G/H.*

Proof. Remark that K acts transitively on the base G/P of the fibration $f : G/H \to G/P$, since P is parabolic. We are then reduced to finding a fundamental domain for the action of $K \cap P = K \cap L$ on the fiber $L/L \cap H$.

Flensted-Jensen proves in [FJ80, Section 2] that a fundamental domain is given by the positive Weyl chamber of a root system which is in general different from the restricted root system described above. However, in our situation, the group L and the involution σ are complex, and this allow to show that the two chambers are the same.

More precisely, Flensted-Jensen considers the subspace $\mathfrak{l}'$ of elements fixed by the involution $\sigma\theta$. The positive Weyl chamber he considers is then a positive Weyl chamber for the root system formed by the non zero eigenvalues of the action of $\mathrm{ad}(\mathfrak{a}_s)$ on $\mathfrak{l}'$. Now remark that the involution $\sigma\theta$ stabilizes any of the subspaces $\bar{\mathfrak{l}}_{\bar{\alpha}/2}$, which we may decompose as $\bar{\mathfrak{l}}_{\bar{\alpha}/2} = \bar{\mathfrak{l}}'_{\bar{\alpha}/2} \oplus \bar{\mathfrak{l}}''_{\bar{\alpha}/2}$ where $\bar{\mathfrak{l}}'_{\bar{\alpha}/2} = \bar{\mathfrak{l}}_{\bar{\alpha}/2} \cap \mathfrak{l}'$ and $\bar{\mathfrak{l}}''_{\bar{\alpha}/2}$ is the subspace of elements x such that $\sigma\theta(x) = -x$. Furthermore, since $\sigma(it) = i\sigma(t)$, multiplication by i induces a bijection between $\bar{\mathfrak{l}}'_{\bar{\alpha}/2}$ and $\bar{\mathfrak{l}}''_{\bar{\alpha}/2}$, and in particular $\bar{\mathfrak{l}}'_{\bar{\alpha}/2}$ is not $\{0\}$ if and only if so is $\bar{\mathfrak{l}}_{\bar{\alpha}/2}$. As a consequence, the set of non zero eigenvalues of the action of $\mathrm{ad}(\mathfrak{a}_s)$ on $\mathfrak{l}'$ is precisely $\bar{\Phi}$.

The reader may find a more detailed account of the results of Flensted-Jensen and of the structure of the action of K on G/H in [vdB05, Section 3]. □ □

2.4. **Colored data for horosymmetric spaces.** As a parabolic induction from a symmetric space, H is a *spherical subgroup* of G, that is, any Borel subgroup of G acts with an open dense orbit on G/H (see [Bri, Per14, Tim11, Kno91] for general presentations of spherical homogeneous spaces, and spherical varieties which will appear later).

Given a choice of Borel subgroup B, a spherical homogeneous space G/H is determined by three combinatorial objects (the highly non-trivial theorem that these objects fully determine H up to conjugacy was obtained by Losev [Los09]).

- The first one is its associated lattice $\mathcal{M}$, defined as the subgroup of characters $\chi \in \mathfrak{X}(B)$ such that there exists a function $f \in \mathbb{C}(G/H)$ with $b \cdot f = \chi(b)f$ for all $b \in B$ (where $b \cdot f(x) = f(b^{-1}x)$ by definition). Let us call $\mathcal{M}$ the *spherical lattice* of G/H. Let $\mathcal{N} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathbb{Z})$ denote the dual lattice.
- The second one, the *valuation cone* $\mathcal{V}$, is defined as the set of elements of $\mathcal{N} \otimes \mathbb{Q}$ which are induced by the restriction of G-invariant, $\mathbb{Q}$-valued valuations on $\mathbb{C}(G/H)$ to B-semi-invariant functions as in the definition of $\mathcal{M}$.
- Finally, the third object needed to characterize the spherical homogeneous space G/H is the *color map* $\rho : \mathcal{D} \to \mathcal{N}$, as a map from an abstract finite set $\mathcal{D}$ to $\mathcal{N}$, that is, we only need to know the image of ρ and the cardinality of its fibers. The set $\mathcal{D}$ is actually the set of codimension one B-orbits in G/H, called *colors*, and the map ρ is obtained by associating to a color D the element of $\mathcal{N}$ induced by the divisorial valuation on $\mathbb{C}(G/H)$ defined by D.

In the case of horosymmetric spaces (which are parabolic inductions from symmetric spaces) these data may mostly be interpreted in terms of the restricted root system for a well chosen Borel B.

Let Q be the parabolic subgroup of G opposite to P with respect to L, that is, the only parabolic subgroup of G such that $Q \cap P = L$ and L is also a Levi subgroup of Q. Let B be a Borel subgroup such that $T \subset B \subset Q$. Then $B \cap L$ is a Borel subgroup of L. By [DP83, Lemma 1.2], we may choose a Borel subgroup of L, containing T, or equivalently a positive root system Φ_L^+ in Φ_L, so that for any positive root $\alpha \in \Phi_L^+$, either $\sigma(\alpha) = \alpha$ or $-\sigma(\alpha)$ is in Φ_L^+. Since Borel subgroups of L containing T are conjugate by an element of $N_L(T)$, we may choose a conjugate of the initially chosen Borel subgroup of G so that the resulting Borel B satisfies $T \subset B \subset Q$ and that $B \cap L$ satisfies the above property with respect to σ.

We fix such a Borel subgroup and denote by Φ^+ the corresponding positive root system of Φ. We will use the notations $\Phi_L^+ = \Phi^+ \cap \Phi_L$ and $\Phi_s^+ := \Phi_L^+ \cap \Phi_s$. Note also that $\Phi_{P^u} = -\Phi^+ \setminus \Phi_L$ and $\Phi_{Q^u} = -\Phi_{P^u}$. Let S denote the set of *simple roots* of Φ generating Φ^+, and let $S_L = \Phi_L \cap S$, $S_s = \Phi_s \cap S$. This induces a natural choice of simple roots in the restricted root system: $\bar{S} = \{\bar{\alpha}; \alpha \in S_s\}$, and corresponding positive roots $\bar{\Phi}^+ = \{\bar{\alpha}; \alpha \in \Phi_s^+\}$.

Given $\alpha \in \Phi$, recall that the *coroot* $\alpha^\vee$ is defined as the unique element in $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{t}$ such that for all $x \in \mathfrak{t}$, $\alpha(x) = 2\kappa(x, \alpha^\vee)/\kappa(\alpha^\vee, \alpha^\vee)$ where κ denotes the Killing form on $\mathfrak{g}$. Since α is real on $\mathfrak{t} \cap i\mathfrak{k}$, the coroot $\alpha^\vee$ is in $\mathfrak{a} = \mathfrak{t} \cap i\mathfrak{k}$ which we may also identify with $\mathfrak{Y}(T) \otimes \mathbb{R}$.

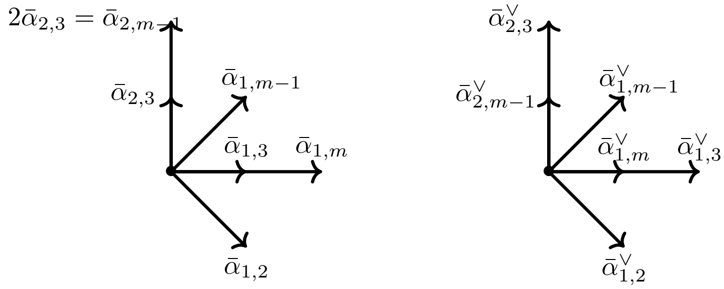
Example 2.21. In our favorite group SL_n, the coroot $\alpha_{j,k}^\vee$ is the diagonal matrix with l^{th}-coefficient equal to $\delta_{l,j} - \delta_{l,k}$.

We use also the notion of restricted coroots for the restricted root, as defined in [Vus90, Section 2.3].

Definition 2.22. Given $\alpha \in \Phi_s$ the *restricted coroot* $\bar{\alpha}^\vee$ is defined as:

- $\alpha^\vee/2$ if $-\sigma(\alpha) = \alpha$ (α is then called a real root),

Figure 1. Roots and coroots of type AIII(2, $m > 4$)



- $(\alpha^\vee - \sigma(\alpha^\vee))/2 = (\alpha^\vee + (-\sigma(\alpha))^\vee)/2$ if $\sigma(\alpha)(\alpha^\vee) = 0$,
- $(\alpha - \sigma(\alpha))^\vee$ if $\sigma(\alpha)(\alpha^\vee) = 1$, in which case $\alpha - \sigma(\alpha) \in \Phi_s$.

The restricted coroots form a root system dual to the restricted root system, and we thus call *simple restricted coroots* the basis of this root system corresponding to the choice of positive roots $\bar{\alpha}^\vee$ for $\alpha \in \Phi^+$. One has to be careful here: in general the simple restricted corots are not the coroots of simple restricted roots.

Example 2.23. Consider the example of type AIII(2, $m > 4$). Then we already described the restricted root system in Example 2.19. There are two real roots $\alpha_{1,m}$ and $\alpha_{2,m-1}$. The restricted coroots are diagonal matrices of the form

$$\mathrm{diag}(b_1, b_2, 0, \dots, 0, -b_2, -b_1)$$

and we write this more concisely as a point with coordinates (b_1, b_2). The restricted coroot $\bar{\alpha}^\vee_{1,m}$ is then $(1/2, 0)$, while $\bar{\alpha}^\vee_{2,m-1} = (0, 1/2)$. The roots $\alpha_{1,2}$ and $\alpha_{1,m-1}$ satisfy $\sigma(\alpha)(\alpha^\vee) = 0$, hence we have $\bar{\alpha}^\vee_{1,2} = (1/2, -1/2)$ and $\bar{\alpha}^\vee_{1,m-1} = (1/2, 1/2)$. Finally, the roots $\alpha_{1,3}$ and $\alpha_{2,3}$ satisfy $\sigma(\alpha)(\alpha^\vee) = 1$, and we have $\alpha_{1,3} - \sigma(\alpha_{1,3}) = \alpha_{1,m}$ and $\alpha_{2,3} - \sigma(\alpha_{2,3}) = \alpha_{2,m-1}$, hence $\bar{\alpha}^\vee_{1,3} = (1, 0)$ and $\bar{\alpha}^\vee_{2,3} = (0, 1)$. We have described that way all (positive) restricted coroots. Figure 1 illustrates the positive restricted roots and coroots in this example.

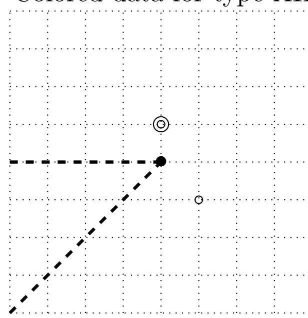
Recall that $f : G/H \to G/P$ denotes the fibration map. The set of colors $\mathcal{D}(G/P)$ of the generalized flag manifold G/P is in bijection with the set $\Phi_{Q^u} \cap S$ of simple roots that are also roots of Q^u, and any pre-image by f of a color of G/P is a color of G/H. Denote by D_α the color of G/P associated with the root $\alpha \in \Phi_{Q^u} \cap S$.

We identify $\mathfrak{a}_s$ and $\mathfrak{Y}(T/T \cap H) \otimes \mathbb{R}$.

Proposition 2.24 ([Tim11, Proposition 20.4] and [Vus90])**.** *Assume that G/H is a horosymmetric space. Then*

- *the spherical lattice $\mathcal{M}$ is the lattice $\mathfrak{X}(T/T \cap H)$,*
- *the valuation cone $\mathcal{V}$ is the negative restricted Weyl chamber $-\mathfrak{a}_s^+$,*
- *the set of colors may be decomposed as a union of two sets $\mathcal{D} = \mathcal{D}(L/L \cap H) \cup f^{-1}\mathcal{D}(G/P)$. The image of the color $f^{-1}(D_\alpha)$ by ρ is the restriction $\alpha^\vee|_{\mathcal{M}}$ of the coroot $\alpha^\vee$ for $\alpha \in \Phi_{Q^u} \cap S$. The image $\rho(\mathcal{D}(L/L \cap H)$ on the other hand is the set of simple restricted coroots.*

Figure 2. Colored data for type AIII(2, $m > 4$)



Remark 2.25. If $G = L$ is semisimple and simply connected, then $\mathcal{M}$ is a lattice between the lattice of restricted weights and the lattice of restricted roots determined by the restricted root system [Vus90]. More precisely, it is the lattice of restricted weights if and only if $H = G^\sigma$ and it is the lattice of restricted roots if and only if $H = N_G(G^\sigma)$.

Remark that the proposition does not give here a complete description of ρ in general as it does not give the cardinality of all orbits. There is however a rather general case where the discussion is simply settled. Say that the symmetric space $L/L \cap H$ has no Hermitian factor if $[L, L] \cap Z_L(L \cap H)$ is finite. Then Vust proved the following full caracterization of ρ:

Proposition 2.26 ([Vus90])**.** *Assume that $L/L \cap H$ has no Hermitian factor. Then the color map ρ is injective on $\mathcal{D}(L/L \cap H)$.*

Note, and this is a general fact for parabolic inductions, that the images of colors in $f^{-1}\mathcal{D}(G/P)$ by ρ all lie in the valuation cone $\mathcal{V}$. Indeed, for any two simple roots α and β, $\kappa(\alpha, \beta) \leq 0$. Given $\alpha \in \Phi_{Q^u} \cap S$, this implies that $\kappa(\alpha, \beta) \leq 0$ for any $\beta \in \Phi_L^+$ and thus $\kappa(\alpha, \bar{\beta}) \leq 0$ for $\beta \in S_s$.

Example 2.27. We draw here Figure 2 as an example of colored data for the symmetric space of type AIII(2, $m > 4$). The dotted grid represents the dual of the spherical lattice (which coincides here with the lattice generated by restricted coroots), the cone delimited by the dashed rays represents the valuation cone (the negative restricted Weyl chamber), and the circles are centered on the points in the image of the color map (the simple restricted coroots), the number of circles reflecting the cardinality of the fiber.

3. Curvature forms

We now begin the study of Kähler geometry on horosymmetric spaces. We first recall how linearized line bundles on homogeneous spaces are encoded by their isotropy character, then we consider K-invariant hermitian metrics. We associate two functions to a hermitian metric: the quasipotential and the toric potential. We express the curvature form of the metric in terms of the isotropy character and toric potential, using the quasipotential as a tool in the proof.

For this section, we use the letter q to denote a metric, as the letter h denotes elements of the group H. Recall that given a hermitian metric q on a line bundle $\mathcal{L}$, its curvature form ω may be defined locally as follows. Let s be a local trivialization of $\mathcal{L}$ and let φ denote the function defined by $\varphi = -\ln |s|_q^2$. Then the curvature form is the globally defined form which satisfies locally $\omega = i\partial\bar{\partial}\varphi$.

3.1. **Linearized line bundles on horosymmetric homogeneous spaces.** Let $\mathcal{L}$ be a G-linearized line bundle on G/H. The pulled back line bundle $\pi^*\mathcal{L}$ on G is trivial, and we denote by s a G-equivariant trivialization of $\pi^*\mathcal{L}$ on G. Denote by χ the character of H defined by $h\cdot\xi = \chi(h)\xi$ for any ξ in the fiber $\mathcal{L}_{eH}$. It fully determines the G-linearized line bundle $\mathcal{L}$. The line bundle is trivializable on G/H if and only if χ is the restriction of a character of G.

Example 3.1. The anticanonical line bundle admits a natural linearization, induced by the linearization of the tangent bundle. We may determine the isotropy character χ from the isotropy representation of H on the tangent space at eH. If one identify this tangent space with $\mathfrak{g}/\mathfrak{h}$, then, working at the level of Lie algebras, the isotropy representation is given by $h\cdot(\xi+\mathfrak{h}) = [h,\xi]+\mathfrak{h}$ for $h\in\mathfrak{h}$, $\xi\in\mathfrak{g}$. Taking the determinant of this representation, we obtain for a horosymmetric homogeneous space G/H that the isotropy Lie algebra character for the anticanonical line bundle is the restriction of the character $\sum_{\alpha\in\Phi_{Q^u}}\alpha$ of $\mathfrak{p}$ to $\mathfrak{h}$.

Example 3.2. On a Hermitian symmetric space, there may be non-trivial line bundles on G/H, as there may exist characters of H which are not restrictions of characters of G. Let us illustrate this with our favorite type AIII example. Consider the matrix

$$M_r = \begin{pmatrix} \frac{1}{\sqrt{2}}I_r & 0 & \frac{1}{\sqrt{2}}S_r \\ 0 & I_{n-2r} & 0 \\ -\frac{1}{\sqrt{2}}S_r & 0 & \frac{1}{\sqrt{2}}I_r \end{pmatrix} \quad \text{so that} \quad M_rJ_rM_r^{-1} = \begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

then $M_rHM_r^{-1} = S(GL_r\times GL_{n-r})$. This group obviously has non trivial characters not induced by a character of the (semisimple) group G, for example $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mapsto \det(A)$. Write an element of H as

$$h = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{21}S_r \\ S_rA_{13}S_r & S_rA_{12} & S_rA_{11}S_r \end{pmatrix}$$

then composing with conjugation by M we obtain the non-trivial character

$$\chi : h \mapsto \det(A_{11} + A_{13}S_r).$$

Example 3.3. Consider the simplest example of type AIII, that is, $\mathbb{P}^1\times\mathbb{P}^1\setminus\operatorname{diag}(\mathbb{P}^1)$ equipped with the diagonal action of SL_2, and with base point $([1:1],[-1:1])$. Then we have naturally linearized line bundles given by the restriction of $\mathcal{O}(k,m)$ for $k,m\in\mathbb{N}$. The character associated to the line bundle $\mathcal{O}(k,m)$ is χ^{k-m} with χ as above, which translates here as $\chi : \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \mapsto a+b$. In particular we recover that it is trivial if and only if $k=m$.

3.2. **Quasipotential and toric potential.** Let q be a smooth K-invariant metric on $\mathcal{L}$.

Definition 3.4.

- The *quasipotential* of q is the function ϕ on G defined by

$$\phi(g) = -2\ln|s(g)|_{\pi^*q}.$$

- The *toric potential* of q is the function $u : \mathfrak{a}_s \to \mathbb{R}$ defined by
$$u(x) = \phi(\exp(x)).$$

Proposition 3.5. *The function ϕ satisfies the following equivariance relation:*
$$\phi(kgh) = \phi(g) - 2\ln|\chi(h)|.$$
In particular ϕ is fully determined by u.

Proof. First, by G-invariance of s, we have
$$\begin{aligned}\phi(kgh) &= -2\ln|kgh \cdot s(e)|_{\pi^*q}\\ &= -2\ln|k\cdot g\cdot h\cdot \pi_* s(e)|_q\end{aligned}$$
by equivariance of π
$$= -2\ln|g\cdot\chi(h)\pi_*(s(e))|_q$$
by K-invariance of q and by definition of χ
$$= -2\ln|g\cdot s(e)|_{\pi^*q} - 2\ln|\chi(h)|$$
Hence the equivariance relation.

Recall from Proposition 2.20 that any K-orbit on G/H intersects the image of $\mathfrak{a}_s$, so in view of the equivariance formula for ϕ, we see that ϕ, hence q, is fully determined by u. □ □

3.3. **Reference $(1,0)$-forms.** We choose root vectors $0 \neq e_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Phi$ such that $[\theta(e_\alpha), e_\alpha] = \alpha^\vee$. Using these root vectors, we can give a more explicit decomposition of $\mathfrak{h}$:
$$\mathfrak{h} = \bigoplus_{\alpha\in\Phi_{P^u}} \mathbb{C}e_\alpha \oplus \mathfrak{t}^\sigma \oplus \bigoplus_{\alpha\in\Phi_L^\sigma} \mathbb{C}e_\alpha \oplus \bigoplus_{\alpha\in\Phi_s^+} \mathbb{C}(e_\alpha + \sigma(e_\alpha))$$

Choose a basis $(l_1, \dots, l_r)$ of the real vector space $\mathfrak{a}_s$. Let us further add the vectors e_α for $\alpha \in \Phi_{Q^u}$ and $\tau_\beta = e_\beta - \sigma(e_\beta)$, for $\beta \in \Phi_s^+$. Then we obtain a family which is the complex basis of a complement of $\mathfrak{h}$ in $\mathfrak{g}$. This also defines local coordinates
$$g \exp\left(\sum_j z_j l_j + \sum_\alpha z_\alpha e_\alpha + \sum_\beta z_\beta \tau_\beta\right) H$$
near a point gH in G/H, depending on the choice of g. Let $\gamma_\diamondsuit^g$ denote the element of $\Omega_{gH}^{(1,0)} G/H$ defined by these coordinates, where $\diamondsuit$ is either some j, some α or some β. Then $x \mapsto \gamma_\diamondsuit^{\exp(x)}$ provides a well defined (by Proposition 2.20) $\exp(\mathfrak{a}_s)$-invariant smooth $(1,0)$-form on $\exp(\mathfrak{a}_s)H/H$. From now on we denote by $\gamma_\diamondsuit$ the corresponding $(1,0)$-form and by $\omega_{\diamondsuit,\bar\heartsuit}$ the $(1,1)$-form $i\gamma_\diamondsuit \wedge \bar\gamma_\heartsuit$.

3.4. **Reference volume form and integration.** We introduce a reference volume form on G/H. Recall from example 3.1 that the naturally linearized canonical line bundle $K_{L/L\cap H}$ on the symmetric space $L/L\cap H$ is L-trivial up to passing to a finite tensor power. For simplicity, we ignore this finite tensor power in the following. The general case follows by considering multisections instead of sections. From now on, we assume that there exists a nowhere vanishing section $s_0 : L/L\cap H \to K_{L/L\cap H}$

which is L-equivariant. We can further assume that s_0 coincides with $\bigwedge_j \gamma_j \wedge \bigwedge_\beta \gamma_\beta$ on $\exp(\mathfrak{a}_s)H/H$ where j runs from 1 to r and β runs over the set Φ_s^+.

Recall that f denotes the map $G/H \to G/P$. Let $K_f = K_{G/H} - f^*K_{G/P}$ denote the relative canonical bundle. Then the section s_0 above may be considered as a trivialization of K_f on the fiber above $eP \in G/P$. Since the map f is G-equivariant, K_f admits a natural G-linearization, and we may use the action of the maximal compact group K to build a K-equivariant trivialization s_f of K_f on G/H. Setting $|s_f|_{q_f} = 1$ provides a smooth K-invariant metric q_f on K_f. Let q_P denote the smooth K-invariant metric on $K_{G/P}$ which satisfies $|f_*(\bigwedge_\alpha \gamma_\alpha^e)|_{q_P} = 1$, where α runs over the set Φ_{Q^u}. Pulling it pack provides a smooth K-invariant metric on $f^*K_{G/P}$.

The two metrics together provide a smooth reference metric $q_H = q_f \otimes f^*q_P$ on $K_{G/H} = K_f \otimes f^*K_{G/P}$, which is K-invariant. We denote by dV_H the associated smooth volume form on G/H. It is defined pointwise as follows: if ξ is an element of the fiber of $K_{G/H}$ at gH, then $(dV_H)_{gH} = i^{n^2}|\xi|_{q_H}^{-2}\xi \wedge \bar{\xi}$.

Proposition 3.6. *Let $a \in \mathfrak{a}_s$, then*

$$(dV_H)_{\exp(a)H} = e^{2\sum_\alpha \alpha(a)} \left(\bigwedge_\diamondsuit \omega_{\diamondsuit,\bar{\diamondsuit}} \right)_{\exp(a)H}$$

Proof. At a point $\exp(a)H$ for $a \in \mathfrak{a}_s$, we can choose

$$\xi = \bigwedge_\diamondsuit \gamma_\diamondsuit^{\exp(a)} = \exp(-a)^* \cdot \bigwedge_\diamondsuit \gamma_\diamondsuit^e,$$

and we get

$$\begin{aligned}(dV_H)_{\exp(a)H} &= |\xi|_{q_H}^{-2} i^{n^2} \xi \wedge \bar{\xi} \\ &= |\exp(-a)^* \cdot \bigwedge_\alpha \gamma_\alpha^e|_{f^*q_P}^{-2} i^{n^2} \xi \wedge \bar{\xi}\end{aligned}$$

by definition of q_H and q_f,

$$\begin{aligned}&= |\exp(-a)^* \cdot f_*(\bigwedge_\alpha \gamma_\alpha^e)|_{q_P}^{-2} i^{n^2} \xi \wedge \bar{\xi} \\ &= e^{2\sum_\alpha \alpha(a)} i^{n^2} \xi \wedge \bar{\xi}\end{aligned}$$

because P acts on the fiber at eP of $K_{G/P}$ *via* the character $-\sum_{\alpha \in \Phi_{Q^u}} \alpha$,

$$= e^{2\sum_\alpha \alpha(a)} i^n (-1)^{n(n-1)/2} \xi \wedge \bar{\xi}.$$

□ □

Remark that dV_H depends on the precise choice of basis of the complement of $\mathfrak{h}$ in $\mathfrak{g}$ only by a multiplicative constant, as it only changes the element of the fiber of $K_{G/P}$ at eP where q_P takes value one, and the element of the fiber of K_f at eH where q_f takes value one.

Combining fiber integration with respect to the fibration f, and the formula for integration on symmetric spaces from [FJ80, Theorem 2.6], we obtain a formula that reduces integration of a K-invariant function on G/H with respect to $dV_{G/H}$ to integration of its restriction to $\exp(\mathfrak{a}_s^+)$ with respect to an explicit measure.

Let J_H denote the function on $\mathfrak{a}_s$ defined by

$$J_H(x) = \prod_{\alpha\in\Phi_s^+} |\sinh(2\alpha(x))|.$$

Another possible expression of the function J_H is:

$$J_H(x) = \prod_{\bar\alpha\in\bar\Phi^+} |\sinh(\bar\alpha(x))^{m_{\bar\alpha}}|$$

where $m_{\bar\alpha} = \dim(\bar{l}_{\bar\alpha/2})$ is the number of $\beta\in\Phi_s$ such that $\bar\beta = \bar\alpha$.

Proposition 3.7. *There exists a constant $C_H > 0$ such that for any K-invariant function ψ on G/H which is integrable with respect to dV_H, we have*

$$\int_{G/H} \psi dV_H = C_H \int_{\mathfrak{a}_s^+} \psi(\exp(x)H)J_H(x)dx$$

where dx is a fixed Lebesgue measure on $\mathfrak{a}_s$.

Here again, a more detailed account on the integration formula for symmetric spaces may be found in [vdB05, Section 3].

3.5. **Preparation for curvature form.** To shorten the formulas, we start using the following notations, for $y\in\mathfrak{g}$,

$$\Re(y) = \frac{y-\theta(y)}{2} \in i\mathfrak{k} \qquad \text{and} \qquad \Im(y) = \frac{y+\theta(y)}{2} \in \mathfrak{k}.$$

For $y\in\mathfrak{l}$, we will also use the notations

$$\mathcal{H}(y) = \frac{y+\sigma(y)}{2} \in \mathfrak{h} \qquad \text{and} \qquad \mathcal{P}(y) = \frac{y-\sigma(y)}{2}.$$

Remark that $\tau_\beta = 2\mathcal{P}(e_\beta)$ and define $\mu_\beta = 2\mathcal{H}(e_\beta)$.

Lemma 3.8. *Let $a\in\mathfrak{a}_s$ be such that $\beta(a)\neq 0$ for all $\beta\in\Phi_s$. Consider an element D in $\mathfrak{g}$ and write*

$$D = \sum_{1\leq j\leq r} z_j l_j + \sum_{\alpha\in\Phi_{Q^u}} z_\alpha e_\alpha + \sum_{\beta\in\Phi_s^+} z_\beta \tau_\beta + h$$

where $h\in\mathfrak{h}$, and z_j for $1\leq j\leq r$, z_α for $\alpha\in\Phi_{Q^u}$, and z_β for $\beta\in\Phi_s^+$ denote complex numbers. Then we may write $D = A_D + B_D + C_D$ with $A_D\in\mathrm{Ad}(\exp(-a))(\mathfrak{k})$, $B_D\in\mathfrak{a}_1$ and $C_D\in\mathfrak{h}$ as follows.

$$\begin{aligned}
A_D &= \sum_{1\leq j\leq r} \Im(z_j l_j) + \exp(\mathrm{ad}(-a))\Big\{ \sum_{\beta\in\Phi_s^+} \left(\frac{\Im(z_\beta\tau_\beta)}{\cosh(\beta(a))} - \frac{\Im(z_\beta\mu_\beta)}{\sinh(\beta(a))} \right) \\
&\quad + \sum_{\alpha\in\Phi_{Q^u}} 2e^{\alpha(a)}\Im(z_\alpha e_\alpha) \Big\} \\
B_D &= \sum_{1\leq j\leq r} \Re(z_j l_j) \\
C_D &= h + \sum_{\beta\in\Phi_s^+} \big\{ \tanh(\beta(a))\Re(z_\beta\mu_\beta) + \coth(\beta(a))\Im(z_\beta\mu_\beta) \big\} \\
&\quad + \sum_{\alpha\in\Phi_{Q^u}} -e^{2\alpha(a)}\theta(z_\alpha e_\alpha)
\end{aligned}$$

Proof. This is a straightforward rewriting, using the following relations. For $\alpha \in \Phi_{Q^u}$,

$$\exp(\mathrm{ad}(-a))(z_\alpha e_\alpha + \theta(z_\alpha e_\alpha)) = e^{\alpha(-a)} z_\alpha e_\alpha + e^{-\alpha(-a)}\theta(z_\alpha e_\alpha)$$

where we remark that $z_\alpha e_\alpha + \theta(z_\alpha e_\alpha) \in \mathfrak{k}$ and $\theta(z_\alpha e_\alpha) \in \mathfrak{h}$. For the terms in τ_β, we use the relations

$$\begin{aligned} z_\beta\tau_\beta &= \Re(z_\beta\tau_\beta) + \Im(z_\beta\tau_\beta), \\ \exp(\mathrm{ad}(-a))(\Im(z_\beta\tau_\beta)) &= \cosh(\beta(a))\Im(z_\beta\tau_\beta) - \sinh(\beta(a))\Re(z_\beta\mu_\beta), \\ \exp(\mathrm{ad}(-a))(\Im(z_\beta\mu_\beta)) &= \cosh(\beta(a))\Im(z_\beta\mu_\beta) - \sinh(\beta(a))\Re(z_\beta\tau_\beta). \end{aligned}$$

Note that the relations hold because $a \in \mathfrak{a}_s$, hence $\sigma(\beta)(a) = -\beta(a)$. □ □

Let $a \in \mathfrak{a}_s$ be such that $\beta(a) \neq 0$ for all $\beta \in \Phi_s$, and consider now the function

$$D = D(\underline{z}) = \sum_{1\leq j\leq r} z_j l_j + \sum_{\alpha\in\Phi_P^+} z_\alpha e_\alpha + \sum_{\beta\in\Phi_L^+\setminus\Phi_L^\sigma} z_\beta\tau_\beta,$$

where $\underline{z}$ denotes the tuple obtained by merging the tuples $(z_j)_j$, $(z_\alpha)_\alpha$ and $(z_\beta)_\beta$. Let A_D, B_D, C_D be the elements provided by Lemma 3.8 applied to D. Let

$$E = E(\underline{z}) := ([B_D, D] + [C_D, B_D] + [C_D, D])/2$$

and introduce also A_E, B_E, C_E the elements provided by Lemma 3.8 applied to E.

Lemma 3.9. *For small enough values of $\underline{z}$, we have*

$$\exp(D) = \exp(-a)k\exp(a + y + O)\exp(h),$$

where $O = O(\underline{z}) \in \mathfrak{g}$ is of order strictly higher than two in $\underline{z}$, $k = k(\underline{z}) \in K$, $y = y(\underline{z}) \in \mathfrak{a}_s$, and $h = h(\underline{z}) \in \mathfrak{h}$. Furthermore,

$$y = B_D + B_E$$

and

$$\exp(h) = \exp(C_E)\exp(C_D).$$

Proof. Throughout the proof, O denotes an element of $\mathfrak{g}$ for $\underline{z}$ small enough, of order stricly higher than two in $\underline{z}$, which may change from line to line.

We first write $D = A_D + B_D + C_D$, with $A_D \in \mathrm{Ad}(\exp(-a))(\mathfrak{k})$, $B_D \in \mathfrak{a}_s$ and $C_D \in \mathfrak{h}$ given by Lemma 3.8. Remark that they are all of order one in $\underline{z}$

Using the Baker-Campbell-Hausdorff formula [Hoc65, Theorem X.3.1] twice, we obtain that

$$\exp(-A_D)\exp(D)\exp(-C_D) = \exp(B_D + \frac{1}{2}([C_D, B_D] + [C_D, A_D] + [B_D, A_D]) + O).$$

Writing $A_D = D - B_D - C_D$ we easily check that $\frac{1}{2}([C_D, B_D] + [C_D, A_D] + [B_D, A_D])$ is equal to the E introduced before. We may then decompose again E as $A_E + B_E + C_E$ where $A_E \in \mathrm{Ad}(\exp(-a))(\mathfrak{k})$, $B_E \in \mathfrak{a}_s$ and $C_E \in \mathfrak{h}$ given by Lemma 3.8 and all terms are of order two in $\underline{z}$. Using again the Baker-Campbell-Hausdorff formula, we get

$$\exp(D) = \exp(A_D)\exp(A_E)\exp(B_D + B_E + O)\exp(C_E)\exp(C_D).$$

The lemma is thus proved with a final application of the Baker-Campbell-Hausdorff formula to $\exp(a)\exp(y + O)$. □ □

3.6. **Expression of the curvature form.** Given a function $u : \mathfrak{a}_s \to \mathbb{R}$ we may consider its differential $d_a u \in \mathfrak{a}_s^*$ at a given point $a \in \mathfrak{a}_s$ as an element of $\mathfrak{a}^*$ by setting $d_a u(x) = d_a u \circ \mathcal{P}(x)$ and identifying $\mathfrak{a}_s^*$ with $\mathfrak{X}(T/T\cap H)\otimes\mathbb{R}$.

Let $\mathcal{L}$ be a G-linearized line bundle corresponding to the character χ of H. We also denote by χ the corresponding Lie algebra character $\mathfrak{h}\to\mathbb{C}$. Hoping it will cause no confusion, we will also denote by χ the restriction of χ to $\mathfrak{a}\cap\mathfrak{h}$ and consider it as an element of $\mathfrak{a}^*$ by setting $\chi(x) = \chi\circ\mathcal{H}(x)$ for $x\in\mathfrak{a}$.

Let q be a smooth K-invariant metric on $\mathcal{L}$ with toric potential u, and let ω denote the curvature form of q.

Theorem 3.10. *Let $a\in\mathfrak{a}_s$ be such that $\beta(a)\neq 0$ for all $\beta\in\Phi_s$. Then*

$$\omega_{\exp(a)H} = \sum \Omega_{\diamondsuit,\bar{\heartsuit}}\omega_{\diamondsuit,\bar{\heartsuit}}$$

where the sum runs over the indices j, α, β, and the coefficients are as follows. Let $1\leq j, j_1, j_2\leq r$, α, α_1, $\alpha_2\in\Phi_{Q^u}$, and β, β_1, $\beta_2\in\Phi_s^+$ with $\beta_1\neq\beta_2$ and $\alpha_2-\alpha_1\in\Phi_s$, then

$$\Omega_{j_1,\bar{j_2}} = \frac{1}{4}d^2u(l_{j_1},l_{j_2}), \qquad \Omega_{j,\bar{\beta}} = \beta(l_j)(1-\tanh^2(\beta))\chi(\theta(\mu_\beta)),$$

$$\Omega_{\alpha,\bar{\alpha}} = \frac{-e^{2\alpha}}{2}(du-2\chi)(\alpha^\vee), \qquad \Omega_{\alpha_1,\bar{\alpha}_2} = \frac{2\chi([\theta(e_{\alpha_2}),e_{\alpha_1}])}{e^{-2\alpha_1}+e^{-2\alpha_2}},$$

$$\begin{aligned}\Omega_{\beta_1,\bar{\beta}_2} =& \frac{\tanh(\beta_2-\beta_1)}{2}\left\{\frac{1}{\sinh(2\beta_1)}-\frac{1}{\sinh(2\beta_2)}\right\}\chi([\theta(e_{\beta_2}),e_{\beta_1}]) \\ &+\frac{\tanh(\beta_1+\beta_2)}{2}\left\{\frac{1}{\sinh(2\beta_2)}+\frac{1}{\sinh(2\beta_1)}\right\}\chi([\theta(e_{\beta_2}),\sigma(e_{\beta_1})])\end{aligned}$$

and

$$\Omega_{\beta,\bar{\beta}} = \frac{du(\beta^\vee)}{\sinh(2\beta)} - \frac{2}{\cosh(2\beta)}\chi\circ\Re([\theta\sigma(e_\beta),e_\beta])$$

where all quantities are evaluated at a. Finally, the remaining coefficients except obviously the symmetric of those above are zero.

This very involved description drastically simplifies if the restriction of χ to $L\cap H$ is trivial on $[L,L]\cap H$. It is equivalent to the fact that it coincides with the restriction of a character of L to $L\cap H$, or also to the fact that the corresponding line bundle is trivial on the symmetric fiber $L/L\cap H$. This particular case in fact covers a wealth of examples, as it is the case for any choice of line bundle whenever the symmetric fiber has no Hermitian factor. In the Hermitian case there are still plenty of line bundle which satisfy this extra assumption. A remarkable example is the anticanonical line bundle.

Corollary 3.11. *Assume that the restriction of $\mathcal{L}$ to the symmetric fiber $L/L\cap H$ is trivial. Let $a\in\mathfrak{a}_s$ be such that $\beta(a)\neq 0$ for all $\beta\in\Phi_s$. Then $\omega_{\exp(a)H}$ may compactly be written as*

$$\frac{1}{4}d_a^2u(l_{j_1},l_{j_2})\omega_{j_1,\bar{j_2}} + \frac{-e^{2\alpha}}{2}(d_au-2\chi)(\alpha^\vee)\omega_{\alpha,\bar{\alpha}} + \frac{d_au(\beta^\vee)}{\sinh(2\beta(a))}\omega_{\beta,\bar{\beta}}$$

where it is implicit that the three summands are actually sums over $\{1,\dots,r\}$, Φ_{Q^u} and Φ_s^+ respectively.

Example 3.12. Consider the example of $\mathbb{C}^2 \setminus \{0\}$, viewed as a horospherical space under the natural action of SL_2. Then $\mathfrak{a}_s$ is one-dimensional and we may choose $\Phi_{Q^u} = \{\alpha_{2,1}\}$ (Φ_s^+ is obviously empty). We choose $l_1 = \alpha_{2,1}^\vee$ as basis of $\mathfrak{a}_s$ and consider u as a one real variable function, writing $d_a u = u'(y)\alpha_{2,1}$ for $a = yl_1$, so that $d_a^2(l_1, l_1) = 4u''(y)$. Then since $H = \mathrm{Stab}(1, 0)$ has no characters, we have at $\exp(yl_1)H$,

$$\omega = u''(y)\omega_{1,\bar{1}} + e^{-4y}u'(y)\omega_{\alpha_{2,1},\alpha_{\bar{2},1}}.$$

One major application of this general computation of curvature forms, but not the only one, will be through the *Monge-Ampère operator*, which reads, with respect to the reference volume form, as follows.

Corollary 3.13. *Assume that the restriction of $\mathcal{L}$ to the symmetric fiber $L/L \cap H$ is trivial. Let $a \in \mathfrak{a}_s$ be such that $\beta(a) \neq 0$ for all $\beta \in \Phi_s$. Then at $\exp(a)H$, ω^n/dV_H is equal to*

$$\frac{n!}{2^{2r+|\Phi_{Q^u}|}} \frac{\det(((d_a^2 u)(l_j, l_k))_{j,k})}{J_H(a)} \prod_{\alpha\in\Phi_{Q^u}} (2\chi - d_a u)(\alpha^\vee) \prod_{\beta\in\Phi_s^+} |d_a u(\beta^\vee)|$$

Example 3.14. Consider the example of symmetric space of type AIII(2, $m > 4$). We choose as basis l_1, l_2 the basis dual to $(\bar{\alpha}_{1,2}, \bar{\alpha}_{2,3})$. We write $a = a_1 l_1 + a_2 l_2$ and $d_a u = u_1(a)\bar{\alpha}_1 + u_2(a)\bar{\alpha}_2$. We check easily that $\mathcal{P}(\alpha_{1,2}^\vee) = \bar{\alpha}_{1,2}^\vee$, $\mathcal{P}(\alpha_{1,m-1}^\vee) = \bar{\alpha}_{1,m-1}^\vee$, $\mathcal{P}(\alpha_{1,m}^\vee) = \bar{\alpha}_{1,3}^\vee$, $\mathcal{P}(\alpha_{2,m-1}^\vee) = \bar{\alpha}_{2,3}^\vee$, $\mathcal{P}(\alpha_{1,k}^\vee) = \bar{\alpha}_{1,m}^\vee$ and $\mathcal{P}(\alpha_{2,k}^\vee) = \bar{\alpha}_{2,m-1}^\vee$ for $3 \leq k \leq n-2$. Hence we may compute, under the assumption that χ is zero, that at $\exp(a)H$, ω^n/dV_H is equal to $n!/2$ times

$$\frac{(u_{1,1}u_{2,2} - u_{1,2}^2)(2u_1 - u_2)^2 u_1^{2m-7} u_2^2 (u_2 - u_1)^{2m-7}}{\sinh(a_1)^2 \sinh(a_1+a_2)^{2m-8} \sinh(2a_1+2a_2) \sinh(a_1+2a_2)^2 \sinh(a_2)^{2m-8} \sinh(2a_2)}$$

Let us now illustrate on examples how the other terms in the curvature form may appear.

Example 3.15. Consider again $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}(\mathbb{P}^1)$ equipped with the diagonal action of SL_2, and with the linearized line bundle $\mathcal{O}(k, m)$. Then we can take $\beta = \alpha_{1,2}$, $e_\beta = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $l_1 = \beta^\vee = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We may further consider u as a function of a single variable t by writing $a = \left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right)$ and we get

$$\omega_{\exp(a)H} = \frac{u''(t)}{4}\omega_{1,\bar{1}} + 2(m-k)(1 - \tanh^2(2t))(\omega_{1,\bar{\beta}} + \omega_{\beta,\bar{1}}) + \frac{u'(t)}{\sinh(4t)}\omega_{\beta,\bar{\beta}}$$

Example 3.16. Consider the symmetric space of type AIII(1, 3). It admits the non-trivial character $\chi : (a_{i,j}) \mapsto a_{1,1} + a_{1,3}$. For a metric on the line bundle corresponding to this character, we get for example

$$\mathcal{R}([\theta\sigma(e_{\alpha_{1,3}}), e_{\alpha_{1,3}}] = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$$

hence a non-trivial contribution in $\Omega_{\alpha_{1,3},\bar{\alpha}_{1,3}}$ which is equal to

$$\frac{d_a u(\alpha_{1,3}^\vee)}{\sinh(2\alpha_{1,3}(a))} - \frac{1}{\cosh(2\alpha_{1,3}(a))}.$$

Example 3.17. Consider the symmetric space G/G^σ of type AIII(2, 4). It admits the non-trivial character $\chi : (a_{i,j}) \mapsto a_{1,1} + a_{2,2} + a_{2,3} + a_{1,4}$, at the Lie algebra level and for $(a_{i,j}) \in \mathfrak{h}$. For a metric on the line bundle corresponding to this character, we have for example $[\theta(e_{\alpha_{3,4}}), e_{\alpha_{2,4}}] = e_{\alpha_{2,3}}$, $[\theta(e_{\alpha_{3,4}}), \sigma(e_{\alpha_{2,4}})] = -e_{\alpha_{4,1}}$ and $\chi(e_{\alpha_{2,3}}) = \chi \circ \mathcal{H}(e_{\alpha_{2,3}}) = 1/2$, $\chi(-e_{\alpha_{4,1}}) = \chi \circ \mathcal{H}(-e_{\alpha_{4,1}}) = -1/2$, hence, writing $a = \mathrm{diag}(t_1, t_2, -t_2, -t_1)$ we have

$$\Omega_{\alpha_{2,4},\bar{\alpha}_{3,4}} = \frac{-1}{2\cosh(2t_1)\cosh(2t_2)}.$$

Example 3.18. Consider again example 2.12. Using the same notations, we have a non-trivial character χ which associates $a + b$ to any element of H. In this case, since $[\theta(e_{\alpha_{2,3}}), e_{\alpha_{1,3}}] = e_{\alpha_{1,2}}$, and $\chi(e_{\alpha_{1,2}}) = \chi \circ \mathcal{H}(e_{\alpha_{1,2}}) = 1/2$, we have

$$\Omega_{\alpha_{1,3},\bar{\alpha}_{2,3}} = \frac{1}{2\cosh(2t)}$$

at the point $\exp(\mathrm{diag}(t, -t, 0))H$. We check also that

$$\Omega_{\alpha_{2,3},\bar{\alpha}_{2,3}} = e^{2t}(2 - u'(t))/4.$$

The previous examples show that any of the terms written in Theorem 3.10 may be non-zero.

3.7. **Proof of Theorem 3.10. Step 1**

Recall that π denotes the quotient map $G \to G/H$. By definition of the quasipotential $\phi : G \to \mathbb{R}$ of q, $i\partial\bar{\partial}\phi$ is the curvature form of π^*q. Furthermore, this curvature form coincides with $\pi^*\omega$.

Let $f_\diamondsuit \in \mathfrak{g}$ be any of the elements l_j, e_α or τ_β for $1 \leq j \leq r$, $\alpha \in \Phi_{Q^u}$ or $\beta \in \Phi_s^+$. Identifying $\mathfrak{g}$ with $T_e^{(1,0)}G$, we build a global G-invariant $(1,0)$ holomorphic vector fields $\eta_\diamondsuit$ by setting $(\eta_\diamondsuit)_g = g_* f_\diamondsuit \in T_g^{(1,0)}G$. Then

$$\pi^*\omega_g(\eta_\diamondsuit, \bar{\eta}_\heartsuit) = i\frac{\partial^2}{\partial z_\diamondsuit \partial \bar{z}_\heartsuit}\bigg|_0 \phi(g\exp(z_\diamondsuit f_\diamondsuit + z_\heartsuit f_\heartsuit)).$$

By definition, the set of all direct images $\pi_*\eta_\diamondsuit$ at $\exp(a)$ provides a basis of $T^{(1,0)}_{\exp(a)H}G/H$ which coincides with the dual basis to the basis formed by the $(\gamma_\diamondsuit)_{\exp(a)H}$ in $\Omega^{(1,0)}_{\exp(a)H}G/H$. We thus have

$$\begin{aligned}\Omega_{\diamondsuit,\bar{\heartsuit}} &= -i\omega_{\exp(a)H}(\pi_*\eta_\diamondsuit, \pi_*\bar{\eta}_\heartsuit) \\ &= -i(\pi^*\omega)_{\exp(a)}(\eta_\diamondsuit, \bar{\eta}_\heartsuit) \\ &= \frac{\partial^2}{\partial z_\diamondsuit \partial \bar{z}_\heartsuit}\bigg|_0 \phi(\exp(a)\exp(z_\diamondsuit f_\diamondsuit + z_\heartsuit f_\heartsuit)).\end{aligned}$$

Step 2

Set $D = z_\diamondsuit f_\diamondsuit + z_\heartsuit f_\heartsuit$. Using Lemma 3.9, we write

$$\exp(D) = \exp(-a)k\exp(a + y + O)\exp(h).$$

Then

$$\phi(\exp(a)\exp(D)) = \phi(k\exp(a + y + O)\exp(h))$$

by the equivariance property of the quasipotential (Proposition 3.5), this is

$$= \phi(\exp(a + y + O)) - 2\ln|\chi(\exp(h))|$$

Recall from Lemma 3.9 and the notations introduced before this lemma that $y = B_D + B_E$ and $\exp(h) = \exp(C_E)\exp(C_D)$ where $E = \frac{1}{2}([B_D, D] + [C_D, B_D] + [C_D, D])$ and B_D, C_D, B_E, C_E are provided by Lemma 3.8. Note that

$$\begin{aligned} \ln|\chi(\exp(C_E)\exp(C_D))| &= \ln|\chi(\exp(C_E))| + \ln|\chi(\exp(C_D))| \\ &= \ln|e^{\chi(C_E)}| + \ln|e^{\chi(C_D)}| \end{aligned}$$

where we still denote by χ the Lie algebra character $\mathfrak{h} \to \mathbb{C}$ induced by χ,

$$\begin{aligned} &= \mathrm{Re}(\chi(C_E) + \chi(C_D)) \\ &= \mathrm{Re}(\chi(C_E + C_D)). \end{aligned}$$

We may now write

$$\begin{aligned} \Omega_{\diamondsuit,\bar{\heartsuit}} &= \left.\frac{\partial^2}{\partial z_\diamondsuit \partial \bar{z}_\heartsuit}\right|_0 \phi(\exp(a + B_D + B_E + O)) - 2\ln|\chi(\exp(C_E)\exp(C_D))| \\ &= \left.\frac{\partial^2}{\partial z_\diamondsuit \partial \bar{z}_\heartsuit}\right|_0 \phi(\exp(a + B_D + B_E)) - 2\ln|\chi(\exp(C_E)\exp(C_D))| \\ &= \left.\frac{\partial^2}{\partial z_\diamondsuit \partial \bar{z}_\heartsuit}\right|_0 (u(a + B_D + B_E) - 2\ \mathrm{Re}(\chi(C_E + C_D)). \end{aligned}$$

Note that here the term O denoted terms of order strictly higher than two in $(z_\diamondsuit, z_\heartsuit)$, which become negligeable in our computation. Actually, other terms will be negligible and we will now denote by O a sum of terms (which may change from line to line) each with a factor among $z_\diamondsuit^2$, $\bar{z}_\diamondsuit^2$, $z_\diamondsuit \bar{z}_\diamondsuit$, $z_\heartsuit^2$, $\bar{z}_\heartsuit^2$, $z_\heartsuit \bar{z}_\heartsuit$, $z_\diamondsuit z_\heartsuit$ or $\bar{z}_\diamondsuit \bar{z}_\heartsuit$.

Step 3

The case by case computation follows.

1) Consider the case $D = z_1 l_{j_1} + z_2 l_{j_2}$, then we have $B_D = (z_1 + \bar{z}_1) l_{j_1}/2 + (z_2 + \bar{z}_2) l_{j_2}/2$ and $B_E = C_E = C_D = 0$ hence

$$\Omega_{j_1,\bar{j}_2} = \frac{1}{4} d_a^2 u(l_{j_1}, l_{j_2}).$$

2) Consider the case $D = z_1 e_\alpha + z_2 l_j$. By Lemma 3.8, we have $B_D = \Re(z_2 l_j)$ and $C_D = -e^{2\alpha(a)}\theta(z_1 e_\alpha)$. We now compute $E = ([B_D, D] - [B_D, C_D] + [C_D, D])/2$:

$$\begin{aligned} 2[B_D, D] &= O - z_1 \bar{z}_2 \alpha(\theta(l_j)) e_\alpha, \\ 2[B_D, C_D] &= z_2 \bar{z}_1 \alpha(l_j) e^{2\alpha(a)} \theta(e_\alpha) + O, \\ [C_D, D] &= -z_2 \bar{z}_1 \alpha(l_j) e^{2\alpha(a)} \theta(e_\alpha) + O, \end{aligned}$$

hence

$$E = \frac{1}{4} z_1 \bar{z}_2 \alpha(l_j) e_\alpha - \frac{3}{4} z_2 \bar{z}_1 \alpha(l_j) e^{2\alpha(a)} \theta(e_\alpha) + O.$$

Using Lemma 3.8 again we check that $B_E = O$ is negligible and

$$C_E = -2 z_2 \bar{z}_1 \alpha(l_j) e^{2\alpha(a)} \theta(e_\alpha) + O.$$

Since $\theta(e_\alpha)$ is in the Lie algebra of the unipotent radical of H, we have $\chi(\theta(e_\alpha)) = 0$, we may thus end the computation and obtain

$$\Omega_{j,\bar{\alpha}} = \Omega_{\alpha,\bar{j}} = 0.$$

3) Consider the case $D = z_1 e_{\alpha_1} + z_2 e_{\alpha_2}$. We have $B_D = 0$ and

$$C_D = -e^{2\alpha_1(a)} \theta(z_1 e_{\alpha_1}) - e^{2\alpha_2(a)} \theta(z_2 e_{\alpha_2}).$$

Then $E=[C_D,D]/2$ is equal to

$$E=O-e^{2\alpha_1(a)}\bar{z}_1z_2[\theta(e_{\alpha_1}),e_{\alpha_2}]/2-e^{2\alpha_2(a)}z_1\bar{z}_2[\theta(e_{\alpha_2}),e_{\alpha_1}]/2$$

We then need to treat several cases separately, depending on $\alpha_1-\alpha_2$.

3.i) If $\alpha_1=\alpha_2=\alpha$ then we have

$$E=-\frac{1}{2}e^{2\alpha(a)}(\bar{z}_1z_2+z_1\bar{z}_2)\alpha^\vee+O$$

hence

$$B_E=-\frac{1}{2}e^{2\alpha(a)}(\bar{z}_1z_2+z_1\bar{z}_2)\mathcal{P}(\alpha^\vee)+O$$

and

$$C_E=-\frac{1}{2}e^{2\alpha(a)}(\bar{z}_1z_2+z_1\bar{z}_2)\mathcal{H}(\alpha^\vee)+O.$$

Since χ is trivial on the unipotent radical of H, we have

$$\mathrm{Re}(\chi(C_D+C_E))=\mathrm{Re}(\chi(C_E))=\chi\circ\Re(C_E)=\chi(C_E).$$

We then end the computation to obtain

$$\Omega_{\alpha,\bar{\alpha}}=\frac{-1}{2}e^{2\alpha(a)}(d_au(\alpha^\vee)-2\chi(\alpha^\vee))$$

3.ii) If $\alpha_2-\alpha_1\in\Phi_L^\sigma$ then we get $B_E=O$ and $C_E=O+E$. Furthermore, we check easily that $[\theta(e_{\alpha_1}),e_{\alpha_2}]\in\mathfrak{g}_{\alpha_2-\alpha_1}\subset[\mathfrak{h},\mathfrak{h}]$ (consider $[[\theta(e_{\alpha_2-\alpha_1}),e_{\alpha_2-\alpha_1}],e_{\alpha_2-\alpha_1}]$) so $\chi([\theta(e_{\alpha_1}),e_{\alpha_2}])=0$, and the same holds for $[\theta(e_{\alpha_2}),e_{\alpha_1}]$. We can then end the computation and obtain

$$\Omega_{\alpha_1,\bar{\alpha_2}}=0.$$

3.iii) If $\alpha_2-\alpha_1\in\Phi_L\setminus\Phi_L^\sigma$ then we have $B_E=O$ and

$$\begin{aligned}C_E=&O-e^{2\alpha_1(a)}\bar{z}_1z_2\mathcal{H}([\theta(e_{\alpha_1}),e_{\alpha_2}])/2\\&-e^{2\alpha_2(a)}z_1\bar{z}_2\mathcal{H}([\theta(e_{\alpha_2}),e_{\alpha_1}])/2\\&+\tanh((\alpha_2-\alpha_1)(a))\Re(-e^{2\alpha_1(a)}\bar{z}_1z_2\mathcal{H}([\theta(e_{\alpha_1}),e_{\alpha_2}])/2))\\&+\coth((\alpha_2-\alpha_1)(a))\Im(-e^{2\alpha_1(a)}\bar{z}_1z_2\mathcal{H}([\theta(e_{\alpha_1}),e_{\alpha_2}])/2))\\&+\tanh((\alpha_1-\alpha_2)(a))\Re(-e^{2\alpha_2(a)}\bar{z}_2z_1\mathcal{H}([\theta(e_{\alpha_2}),e_{\alpha_1}])/2))\\&+\coth((\alpha_1-\alpha_2)(a))\Im(-e^{2\alpha_2(a)}\bar{z}_2z_1\mathcal{H}([\theta(e_{\alpha_2}),e_{\alpha_1}])/2)).\end{aligned}$$

As a consequence,

$$\begin{aligned}\Re(C_E)=&O-e^{2\alpha_1(a)}\Re(\bar{z}_1z_2\mathcal{H}([\theta(e_{\alpha_1}),e_{\alpha_2}]))/2\\&-e^{2\alpha_2(a)}\Re(z_1\bar{z}_2\mathcal{H}([\theta(e_{\alpha_2}),e_{\alpha_1}]))/2\\&+\tanh((\alpha_2-\alpha_1)(a))\Re(-e^{2\alpha_1(a)}\bar{z}_1z_2\mathcal{H}([\theta(e_{\alpha_1}),e_{\alpha_2}])/2))\\&+\tanh((\alpha_1-\alpha_2)(a))\Re(-e^{2\alpha_2(a)}\bar{z}_2z_1\mathcal{H}([\theta(e_{\alpha_2}),e_{\alpha_1}])/2)).\end{aligned}$$

We then check by computation that

$$\begin{aligned}\Omega_{\alpha_1,\bar{\alpha}_2}&=\frac{1}{2}\Big(e^{2\alpha_2(a)}(1+\tanh((\alpha_1-\alpha_2)(a)))\\&\qquad+e^{2\alpha_1(a)}(1+\tanh((\alpha_2-\alpha_1)(a)))\Big)\chi\circ\mathcal{H}([\theta(e_{\alpha_2}),e_{\alpha_1}])\\&=\frac{2\chi([\theta(e_{\alpha_2},e_{\alpha_1}])}{(e^{-2\alpha_1(a)}+e^{-2\alpha_2(a)})}.\end{aligned}$$

3.iv) If $\alpha_2 - \alpha_1 \in \Phi \setminus \Phi_L$, say $\alpha_1 - \alpha_2 \in \Phi_{P^u}$ for example, then $B_E = O$ and

$$\begin{aligned} C_E = O + &\frac{-e^{2\alpha_2(a)}}{2}\bar{z}_2 z_1[\theta(e_{\alpha_2}), e_{\alpha_1}] \\ &- e^{2(\alpha_1-\alpha_2)(a)}\theta(\frac{-e^{2\alpha_1(a)}}{2}\bar{z}_1 z_2[\theta(e_{\alpha_1}), e_{\alpha_2}]). \end{aligned}$$

Since $[\theta(e_{\alpha_2}), e_{\alpha_1}]$ is in the Lie algebra of the unipotent radical of H we have the vanishing $\chi([\theta(e_{\alpha_2}), e_{\alpha_1}]) = 0$ hence

$$\Omega_{\alpha_1,\bar{\alpha}_2} = 0.$$

3.v) Finally, if $\alpha_2 - \alpha_1 \notin \Phi$, then we have $[\theta(e_{\alpha_1}), e_{\alpha_2}] = [\theta(e_{\alpha_2}), e_{\alpha_1}] = 0$ hence $B_E = O$ and $C_E = O$, and we deduce

$$\Omega_{\alpha_1,\bar{\alpha}_2} = 0.$$

4) Consider now the case $D = z_1 l_j + z_2\tau_\beta$. Then $B_D = \Re(z_1 l_j)$ and

$$C_D = \tanh(\beta(a))\Re(z_2\mu_\beta) + \coth(\beta(a))\Im(z_2\mu_\beta).$$

We compute

$$[B_D, D] = O + \bar{z}_1 z_2 \beta(l_j)\mu_\beta/2$$

$$\begin{aligned} [C_D, B_D] =&O + \frac{z_1\bar{z}_2}{4}\beta(l_j)(\coth(\beta(a)) - \tanh(\beta(a)))\theta(\tau_\beta) \\ &- \frac{\bar{z}_1 z_2}{4}\beta(l_j)(\tanh(\beta(a)) + \coth(\beta(a))\tau_\beta \end{aligned}$$

and

$$[C_D, D] = O + \frac{z_1\bar{z}_2}{2}\beta(l_j)(\coth(\beta(a) - \tanh(\beta(a))\theta(\tau_\beta).$$

From these computations we deduce

$$\begin{aligned} E =&O + \bar{z}_1 z_2\frac{\beta(l_j)}{2}(\mu_\beta - \frac{\tanh(\beta(a) + \coth(\beta(a))}{2}\tau_\beta) \\ &+ z_1\bar{z}_2\frac{3\beta(l_j)}{4}(\coth(\beta(a)) - \tanh(\beta(a)))\theta(\tau_\beta)). \end{aligned}$$

We then have $B_E = O$ and

$$\begin{aligned} \Re(C_E) = O + &\frac{\beta(l_j)}{2}\Re(\bar{z}_1 z_2\mu_\beta) \\ &+ \tanh(\beta(a))\Re(\bar{z}_1 z_2\frac{-\beta(l_j)}{4}(\tanh(\beta(a)) + \coth(\beta(a)))\mu_\beta) \\ &+ \tanh(-\beta(a))\Re(z_1\bar{z}_2\frac{3\beta(l_j)}{4}(\coth(\beta(a)) - \tanh(\beta(a)))\theta(\mu_\beta)) \\ = O + &\beta(l_j)(1 - \tanh^2(\beta(a)))\Re(\bar{z}_1 z_2\mu_\beta) \end{aligned}$$

since $\Re(z_1\bar{z}_2\theta(\mu_\beta)) = -\Re(\theta(z_1\bar{z}_2\theta(\mu_\beta)) = -\Re(\bar{z}_1 z_2\mu_\beta)$. We may thus finish the computation to obtain

$$\Omega_{j,\bar{\beta}} = \beta(l_j)(1 - \tanh^2(\beta(a)))\chi(\theta(\mu_\beta)).$$

5) Consider the case $D = z_1\tau_\beta + z_2 e_\alpha$. Then we have $B_D = 0$ and

$$C_D = -e^{2\alpha(a)}\theta(z_2 e_\alpha) + \tanh(\beta(a))\Re(z_1\mu_\beta) + \coth(\beta(a))\Im(z_1\mu_\beta).$$

Then $E = [C_D, D]/2$, which is equal to

$$\frac{z_1\bar{z}_2}{2}e^{2\alpha(a)}[\tau_\beta, \theta(e_\alpha)] + \frac{-\bar{z}_1 z_2}{4}(\coth(\beta(a)) - \tanh(\beta(a)))[e_\alpha, \theta(\mu_\beta)] + O.$$

We then remark that $[\tau_\beta, \theta(e_\alpha)] \in \mathfrak{g}_{-\alpha+\beta} \oplus \mathfrak{g}_{-\alpha+\sigma(\beta)}$ and $[e_\alpha, \theta(\mu_\beta)] \in \mathfrak{g}_{\alpha-\beta} \oplus \mathfrak{g}_{\alpha-\sigma(\beta)}$. It is impossible for $-\alpha+\beta$ as well as for $-\alpha+\sigma(\beta)$ to lie in Φ_L (write these roots in a basis of simple roots adapted to P to check this assertion). Hence we obtain that C_E is a sum of a negligible term and a term in $\mathfrak{g}_{-\alpha+\beta} \oplus \mathfrak{g}_{-\alpha+\sigma(\beta)} \oplus \mathfrak{g}_{\alpha-\beta)} \oplus \mathfrak{g}_{\alpha-\sigma(\beta)} \cap \mathfrak{h}$. Since this last space is contained in the Lie algebra of the unipotent radical of H, we obtain that $\chi \circ \Re(C_E)$ is negligible, hence

$$\Omega_{\beta,\bar{\alpha}} = 0.$$

6) Consider finally the case $D = z_1\tau_{\beta_1} + z_2\tau_{\beta_2}$. Then we have $B_D = 0$ and

$$\begin{aligned} C_D =& \tanh(\beta_1(a))\Re(z_1\mu_{\beta_1}) + \coth(\beta_1(a))\Im(z_1\mu_{\beta_1}) \\ &+ \tanh(\beta_2(a))\Re(z_2\mu_{\beta_2}) + \coth(\beta_2(a))\Im(z_2\mu_{\beta_2}). \end{aligned}$$

Then in view of the relation $\coth(x) - \tanh(x) = 2/\sinh(2x)$, we have

$$E = \frac{\bar{z}_2 z_1[\theta(\mu_{\beta_2}), \tau_{\beta_1}]}{2\sinh(2\beta_2(a))} + \frac{\bar{z}_1 z_2[\theta(\mu_{\beta_1}), \tau_{\beta_2}]}{2\sinh(2\beta_1(a))} + O$$

We separate in two cases the end of the computation.

6.i) If $\beta_1 \neq \beta_2$ then, note that for $1 \leq j \neq k \leq 2$, we have

$$[\theta(\mu_{\beta_k}), \tau_{\beta_j}] = 2\mathcal{P}([\theta(e_{\beta_k}), e_{\beta_j}]) + 2\mathcal{P}([\theta\sigma(e_{\beta_k}), e_{\beta_j}])$$

where $[e_{\beta_j}, \theta(e_{\beta_k})] \in \mathfrak{g}_{\beta_j-\beta_k}$ and $[e_{\beta_j}, \theta\sigma(e_{\beta_k})] \in \mathfrak{g}_{\beta_j-\sigma(\beta_k)}$. It shows that B_E is negligible, and that $\Re(C_E)$ is equal to the sum of a negligible term and

$$\begin{aligned} &\frac{\tanh(\beta_2(a)-\beta_1(a))}{\sinh(2\beta_1(a))}\Re(\bar{z}_1 z_2\mathcal{H}([\theta(e_{\beta_1}), e_{\beta_2}])) \\ &+ \frac{\tanh(\beta_2(a)-\sigma(\beta_1)(a))}{\sinh(2\beta_1(a))}\Re(\bar{z}_1 z_2\mathcal{H}([\theta\sigma(e_{\beta_1}), e_{\beta_2}])) \\ &+ \frac{\tanh(\beta_1(a)-\beta_2(a))}{\sinh(2\beta_2(a))}\Re(\bar{z}_2 z_1\mathcal{H}([\theta(e_{\beta_2}), e_{\beta_1}])) \\ &+ \frac{\tanh(\beta_1(a)-\sigma(\beta_2)(a))}{\sinh(2\beta_2(a))}\Re(\bar{z}_2 z_1\mathcal{H}([\theta\sigma(e_{\beta_2}), e_{\beta_1}])). \end{aligned}$$

We may rewrite this as

$$\begin{aligned} &\tanh(\beta_2-\beta_1)(\frac{1}{\sinh(2\beta_1(a)} - \frac{1}{\sinh(2\beta_2(a)})\Re(\bar{z}_1 z_2\mathcal{H}([\theta(e_{\beta_1}), e_{\beta_2}])) \\ &+ \tanh(\beta_1+\beta_2)(\frac{1}{\sinh(2\beta_1(a)} + \frac{1}{\sinh(2\beta_2(a)})\Re(\bar{z}_1 z_2\mathcal{H}([\theta\sigma(e_{\beta_1}), e_{\beta_2}])). \end{aligned}$$

We compute now

$$\begin{aligned} \Omega_{\beta_1,\bar{\beta}_2} =& \frac{1}{2}\tanh(\beta_2(a)-\beta_1(a))(\frac{1}{\sinh(2\beta_1(a)} - \frac{1}{\sinh(2\beta_2(a)})\chi([\theta(e_{\beta_2}), e_{\beta_1}]) \\ &+ \frac{1}{2}\tanh(\beta_1+\beta_2)(\frac{1}{\sinh(2\beta_1(a)} + \frac{1}{\sinh(2\beta_2(a)})\chi([\theta(e_{\beta_2}), \sigma(e_{\beta_1})]) \end{aligned}$$

6.ii) If $\beta_1 = \beta_2 = \beta$ then we have

$$E = \frac{z_1\bar{z}_2 + \bar{z}_1 z_2}{\sinh(2\beta)}(\mathcal{P}(\beta^\vee) + \mathcal{P}([\theta\sigma(e_\beta), e_\beta]))$$

and thus

$$B_E = \frac{z_1\bar{z}_2 + \bar{z}_1 z_2}{\sinh(2\beta)}\mathcal{P}(\beta^\vee)$$

and

$$\begin{aligned}\Re(C_E) &= \frac{z_1\bar{z}_2+\bar{z}_1z_2}{\sinh(2\beta)}\tanh(\beta-\sigma(\beta))\Re\circ\mathcal{H}([\theta\sigma(e_\beta),e_\beta]) \\ &= \frac{z_1\bar{z}_2+\bar{z}_1z_2}{\cosh(2\beta)}\mathcal{H}\circ\Re([\theta\sigma(e_\beta),e_\beta]).\end{aligned}$$

Hence

$$\Omega_{\beta,\bar{\beta}} = \frac{d_au(\beta^\vee)}{\sinh(2\beta)} - \frac{2}{\cosh(2\beta)}\chi\circ\Re([\theta\sigma(e_\beta),e_\beta]).$$

4. Horosymmetric varieties

We move on to introduce horosymmetric varieties. We provide several examples and present the classification theory inherited from that of spherical varieties. We then check the property that a G-invariant irreducible codimension one subvariety in a horosymmetric variety is still horosymmetric.

4.1. **Definition and examples.**

Definition 4.1. A *horosymmetric variety* X is a normal G-variety such that G acts with an open dense orbit which is a *horosymmetric homogeneous space*.

Example 4.2. By Example 2.6, any *horospherical variety* (see [Pas08]) may be considered as a horosymmetric variety. It includes in particular generalized flag manifolds, toric varieties and homogeneous toric bundles.

Example 4.3. Consider the projective plane $\mathbb{P}^2$ equipped with the action of SL_2 or GL_2 induced by a choice of affine chart $\mathbb{C}^2$ in $\mathbb{P}^2$. There are three orbits under this action: the fixed point $\{0\}$, the open dense orbit $\mathbb{C}^2\setminus\{0\}$ and the projective line at infinity $\mathbb{P}^1$. The GL_2-variety $\mathbb{P}^2$ is hence a horospherical variety by Example 2.7. We may further consider the blow up of $\mathbb{P}^2$ at the fixed point $\{0\}$ and lift the action of GL_2 to check that this blow up is also a horospherical variety. More generally, Hirzebruch surfaces have structures of GL_2-horospherical varieties refining their toric structure.

Assume $G = L$ is semisimple and $H = N_G(G^\sigma)$. Then the *wonderful compactifications* of G/H constructed by De Concini and Procesi [DP83] is a horosymmetric variety. It is a particularly nice compactification of G/H caracterized by the following properties. Let r denote the rank of G/H and set $I = \{1,\dots,r\}$.

Theorem 4.4 ([DP83])**.** *The wonderful compactification X of G/H is the unique smooth G-equivariant compactification of X such that:*

- *G-orbit closures X_J in X are in bijection with subsets $J \subset I$ and*
- *all X_J are smooth and intersect transversely, with $X_J = \bigcap_{j\in J} X_{\{j\}}$.*

Furthermore, for each J, there exists a parabolic subgroup P_J of G, with σ-stable semisimple Levi factor L'_J, and an equivariant fibration $X_J \to G/P_J$ with fiber the wonderful compactification of $L'_J/N_{L'_J}((L'_J)^\sigma)$.

Example 4.5. Consider the symmetric space G/H of type AIII$(2, m > 4)$. Recall from Example 2.11 that $H = N_G(H)$. Using the description of G/H as a dense orbit in the product of Grassmannians $X_0 = \mathrm{Gr}_{2,m}\times\mathrm{Gr}_{m-2,m}$ as in Example 2.10, we obtain a first example of (horo)symmetric variety with open orbit G/H: this product of Grassmannians X_0 itself. It contains three orbits under the action of

$G = \mathrm{SL}_m$: the dense orbit of pairs of linear subspaces in direct sum, the closed orbit of flags (V_1, V_2) with $V_1 \subset V_2$, and the codimension one orbit of pairs (V_1, V_2) with $\dim(V_1 \cap V_2) = 1$. One can blow up X_0 along the closed orbit to obtain another G-equivariant compactification X of G/H. This new compactification X is none other than the wonderful compactification of G/H.

In higher ranks, one may still obtain the wonderful compactifications from the product of Grassmannians, but this requires a more involved sequence of blow-ups.

From Theorem 4.4, one sees the first examples of horosymmetric varieties that are neither horospherical nor symmetric. Indeed, the description of orbits in a wondeful compactification show that orbit closures in wonderful compactifications are all horosymmetric, with the only symmetric being the open orbit and the only horospherical being the closed one (actually a generalized flag manifold).

Since it applies only to $H = N_G(G^\sigma)$, the construction of De Concini and Procesi does not exhaust the compactifications of symmetric spaces satisfying the properties of Theorem 4.4, still called wonderful compactifications. The simplest example of wonderful compactification which is not in the examples studied by De Concini and Procesi is the following.

Example 4.6. Consider the variety $\mathbb{P}^1 \times \mathbb{P}^1$ equipped with the diagonal action of SL_2. There are two orbits under this action: the diagonal embedding of $\mathbb{P}^1$ and its complement. The complement is open dense and isomorphic to the symmetric space SL_2/T where T is a maximal torus of SL_2.

The wonderful compactification constructed by De Concini and Procesi and corresponding to this involution on the other hand is $\mathbb{P}^2$ seen as a compactification of $\mathrm{SL}_2/N_{\mathrm{SL}_2}(T)$ by a adding a quadric.

Example 4.7. Several papers expanded results valid on the wonderful compactification of a symmetric space to so-called *complete symmetric varieties* (see *e.g.* [DP85, Bif90]), that is, smooth G-equivariant compactifications of G/H that dominate the wonderful compactification. Any complete symmetric variety is a horosymmetric variety.

4.2. **Combinatorial description of horosymmetric varieties.**

4.2.1. *Colored fans.* As spherical varieties, horosymmetric varieties with open orbit G/H are classified by colored fans for the spherical homogeneous space G/H, which are defined in terms of the combinatorial data $\mathcal{M}$, $\mathcal{V}$ and $\rho : \mathcal{D} \to \mathcal{N}$ (Recall these were described in Proposition 2.24).

Definition 4.8.

- A *colored cone* is a pair $(\mathcal{C}, \mathcal{R})$, where $\mathcal{R} \subset \mathcal{D}$, $0 \notin \rho(\mathcal{R})$, and $\mathcal{C} \subset \mathcal{N} \otimes \mathbb{Q}$ is a strictly convex cone generated by $\rho(\mathcal{R})$ and finitely many elements of $\mathcal{V}$ such that the intersection of the relative interior of $\mathcal{C}$ with $\mathcal{V}$ is not empty.
- Given two colored cones $(\mathcal{C}, \mathcal{R})$ and $(\mathcal{C}_0, \mathcal{R}_0)$, we say that $(\mathcal{C}_0, \mathcal{R}_0)$ is a *face* of $(\mathcal{C}, \mathcal{R})$ if $\mathcal{C}_0$ is a face of $\mathcal{C}$ and $\mathcal{R}_0 = \mathcal{R} \cap \rho^{-1}(\mathcal{C}_0)$.
- A *colored fan* is a non-empty finite set $\mathcal{F}$ of colored cones such that the face of any colored cone in $\mathcal{F}$ is still in $\mathcal{F}$, and any $v \in \mathcal{V}$ is in the relative interior of at most one cone.

An equivariant *embedding* (X, x) of G/H is the data of a horosymmetric variety X and a base point $x \in X$ such that $\overline{G \cdot x} = X$ and $\mathrm{Stab}_G(x) = H$.

Theorem 4.9 ([Kno91, Theorem 3.3 and Theorem 4.2])**.** *There is a bijection $(X,x) \longmapsto \mathcal{F}_X$ between embeddings of G/H up to G-equivariant isomorphism and colored fans. There is a bijection $Y \mapsto (\mathcal{C}_Y, \mathcal{R}_Y)$ between the orbits of G in X, and the colored cones in $\mathcal{F}_X$. An orbit Y is in the closure of another orbit Z in X if and only if the colored cone $(\mathcal{C}_Z, \mathcal{R}_Z)$ is a face of $(\mathcal{C}_Y, \mathcal{R}_Y)$. The variety X is complete if and only if the support $|\mathcal{F}_X| = \bigcup_{(\mathcal{C},\mathcal{R})\in\mathcal{F}_X} \mathcal{C}$ contains the valuation cone $\mathcal{V}$.*

Example 4.10. Assume G is a semisimple group, and H is a symmetric subgroup of G. Then there is a natural choice of colored fan given by the negative Weyl chamber and its faces. If $H = N_G(H)$ then the corresponding variety is the wonderful compactification of G/H.

More generally if the valuation cone is strictly convex, then the embedding corresponding to the colored fan given by the valuation cone and its faces is called *wonderful* it it is in addition smooth. There are criterions of smoothness for spherical varieties [Bri91], and some simpler criterions for the case of horospherical [Pas08] and symmetric [Ruz11, Section 3] varieties. It would certainly be possible and useful to derive such a simpler criterion for the class of horosymmetric varieties. In the case of toroidal horosymmetric varieties, which are introduced in the next section, the criterion is very simple, as it is the case for toroidal spherical varieties in general.

4.2.2. *Toroidal horosymmetric varieties.* Given an embedding (X, x) of G/H we denote by $\mathcal{F}_X$ its colored fan and we call the elements of $\mathcal{D}(X) = \bigcup_{(\mathcal{C},\mathcal{R})\in\mathcal{F}} \mathcal{R} \subset \mathcal{D}$ the *colors of* X. It should be noted that the set of colors does not depend on the base point x, but $\mathcal{F}_X$ does. We however omit this dependence in the notation.

Definition 4.11. An embedding is *toroidal* if $\mathcal{D}(X)$ is empty, else it is *colored.*

A toroidal horosymmetric variety is globally a parabolic induction from a symmetric variety. More generally, we record the following elementary statement, easily seen by the classification of horosymmetric varieties by colored fan, and the fact that the colored fan of a parabolic induction is the same as the colored of the embedding one starts with.

Proposition 4.12. *A horosymmetric variety with set of colors $\mathcal{D}_X$ is globally a parabolic induction from a symmetric variety if and only if $\mathcal{D}_X \cap f^{-1}\mathcal{D}(G/P) = \emptyset$.*

Example 4.13. The horospherical variety $\mathbb{P}^2$ under the action of SL_2 is not a global parabolic induction (in particular it is not toroidal), but the blow up of $\mathbb{P}^2$ is.

The following result shows that, in a toroidal horosymmetric variety, there is a well identified toric subvariety which will play an important role in later applications.

Proposition 4.14 ([Kno94, Corollary 8.3 and paragraph after Corollary 6.3])**.** *Let (X, x) be a toroidal embedding of the horosymmetric space G/H, with colored fan $\mathcal{F}_X$. Then the closure Z of $T \cdot x$ in X is the $T/T \cap H$-toric variety whose fan (as a spherical variety) consists of the images, by elements of the restricted Weyl group $\bar{W}$, of the cones $\mathcal{C}$ in the colored cone $\mathcal{F}_X$.*

Remark 4.15. We insist here that we obtain the fan of Z as a spherical variety under the action of $T/T \cap H$. It does not exactly coincide in general with the fan of

Z as a toric variety with the classical conventions, but to the opposite of this fan. We refer to Pezzini [Pez10, Section 2] for details, but the short explanation is that a character λ of a torus T may be interpreted as a regular $B = T$-semi-invariant function on T with weight $-\lambda$: $\lambda(b^{-1}t) = (-\lambda)(b)\lambda(t)$. This difference is important to get the correct expression for the asymptotic behavior of metrics in Section 6. This fact was overlooked in previous work of the author, fortunately with no serious consequences.

Example 4.16. Assume X is the wonderful compactification of a symmetric space then the fan of its toric subvariety Z is the fan obtained by considering the collection of all restricted Weyl chambers for $\bar{\Phi}$ and their faces.

Finally, let us mention the criterion of smoothness for toroidal horosymmetric varieties:

Proposition 4.17. *A toroidal horosymmetric embedding (X, x) is smooth if and only if the toric subvariety Z is smooth, that is, if and only if every cone in the colored fan is generated by a subset of a basis of $\mathcal{N}$.*

4.3. **Facets of a horosymmetric variety.**

Definition 4.18. Let X be a horosymmetric variety under the action of G. A *facet* of X is a G-stable irreducible codimension one subvariety in X.

The goal of this section is to prove the following result.

Proposition 4.19. *Let X be a horosymmetric variety under the action of G, then any facet of X is also a horosymmetric variety under the action of G.*

We will actually obtain more precise statements describing the corresponding horosymmetric homogeneous spaces, using [Bri90]. Let us first introduce some terminology.

Definition 4.20. An *elementary embedding* (E, x) of G/H is an embedding such that the complement of G/H in E is a single codimension one orbit of G. Equivalently, it is an embedding whose colored fan consists of a single ray $\mathcal{C}_E \subset -\mathfrak{a}_s^+$ with no colors.

Elementary embeddings are in bijection with indivisible one parameter subgroups in $-\mathfrak{a}_s^+ \cap \mathfrak{Y}(T_s)$ by selecting the only such one parameter subgroup in the ray associated to the elementary embedding. Given an indivisible $\mu \in -\mathfrak{a}_s^+ \cap \mathfrak{Y}(T_s)$ we denote by (E_μ, x) the corresponding elementary embedding. Furthermore $x_\mu := \lim_{z\to 0} \mu(z) \cdot x$ exists in E_μ and is in the open B-orbit of the codimension one G-orbit [BP87, Section 2.10]. We will use [Bri90] to describe the Lie algebra of the isotropy subgroup of x_μ.

We fixed since Section 2 a maximal torus T of G and a Borel subgroup B containing T. Recall that parabolic subgroups containing B are classified by subsets of the set of simple roots S. More precisely, given a subset $I \subset S$, there is a unique parabolic subgroup Q_I of G containing B such that $\Phi_{Q_I^u} \cap S = S \setminus I$. It further contains a unique Levi subgroup L_I containing T, and $\Phi_{L_I} \cap S = I$. We denote by P_I the parabolic subgroup opposite to Q_I with respect to L_I.

The subgroup $B \cap L$ is a Borel subgroup of L containing T, and we have the same correspondence between subsets I of S_L and parabolic subgroups Q_I^L of L containing $B \cap L$. We have the obvious relation $Q_I^L = Q_I \cap L$, and all of these parabolics are contained in $Q_{S_L} = Q$. The Levi subgroup of Q_I^L containing T is none other than L_I.

Given a one parameter subgroup $\mu \in \mathfrak{Y}(T)$, we obtain a subset of S_L by setting $I(\mu) = \{\alpha \in S_L; \alpha(\mu) = 0\}$. Then the Levi subgroup L_I of $Q^L_{I(\mu)}$ containing T coincides with the centralizer $Z_L(\mu)$ of μ. In particular, μ is contained in the radical of the Levi subgroup $L_{I(\mu)}$. Pay attention to the fact that $Z_L(\mu)$ may be different from $Z_G(\mu)$ here.

Take $\mu \in \mathfrak{Y}(T_s)$. Then the restriction of σ to $L_{I(\mu)} = Z_L(\mu)$ is still a well defined involution. Since μ is contained in the radical of $L_{I(\mu)}$, we may choose a σ-stable complement $\mathfrak{m}$ of $\mathbb{C}\mu$ in $\mathfrak{l}_{I(\mu)}$ which contains the derived Lie algebra $[\mathfrak{l}_{I(\mu)}, \mathfrak{l}_{I(\mu)}]$. Define a new involution σ_μ on $L_{I(\mu)}$ by setting, at the level of Lie algebras,

$$\sigma_\mu(z\mu + m) = z\mu + \sigma(m).$$

This is a well defined involution of Lie algebras thanks to our choice of complement $\mathfrak{m}$.

The following proposition provides a more precise version of Proposition 4.19. Indeed, given a facet Y of X, the union of the open G-orbits in X and Y form an elementary embedding of G/H.

Proposition 4.21. *Let $\mu \in \mathfrak{Y}(T_s)$ indivisible. Then the isotropy subgroup H_μ of x_μ is horosymmetric as follows:*

$$\mathfrak{h}_\mu = \mathfrak{p}^u_{I(\mu)} \oplus \mathfrak{l}^{\sigma_\mu}_{I(\mu)}.$$

Proof. An elementary embedding is toroidal, hence, by Proposition 4.12, it is a global parabolic induction $E_L \hookrightarrow E_\mu \to G/P$ where (E_L, x) is the elementary embedding of $L/L \cap H$ associated with the same one parameter subgroup μ. Since $x_\mu \in E_L$, we obtain that $P^u \subset H_\mu \subset P$ and H_μ is determined by $L \cap H_\mu$.

We now use the results of [Bri90] applied to the case of symmetric spaces to obtain a description of $\mathfrak{l} \cap \mathfrak{h}_\mu$. By Proposition 2.4 in [Bri90] and the remarks about the case of symmetric spaces in Section 2.2 of the same paper, we have

$$\mathfrak{l} \cap \mathfrak{h}_\mu = \mathbb{C}\mu \oplus \mathfrak{t}^\sigma \oplus \bigoplus_{\alpha \in \Phi_L^\sigma} \mathbb{C}e_\alpha \oplus \bigoplus_{\alpha \in \Phi_s^+; \bar{\alpha}(\mu) = 0} \mathbb{C}(e_{-\alpha} + \sigma(e_{-\alpha})) \oplus \bigoplus_{\alpha \in \Phi_s^+; \bar{\alpha}(\mu) \neq 0} \mathbb{C}e_{-\alpha}.$$

Since $\mu \in \mathfrak{Y}(T_s)$, we have $\bar{\alpha}(\mu) = 2\alpha(\mu)$, so we may write the above expression as

$$\mathfrak{l} \cap \mathfrak{h}_\mu = \mathfrak{l}^{\sigma_\mu}_{I(\mu)} \oplus \bigoplus_{\alpha \in \Phi_{P^u_{I(\mu)}} \cap \Phi_L} \mathfrak{g}_\alpha$$

Putting both results together, we get

$$\mathfrak{l} = \mathfrak{l}^{\sigma_\mu}_{I(\mu)} \oplus \bigoplus_{\alpha \in \Phi_{P^u_{I(\mu)}}} \mathfrak{g}_\alpha$$

hence the statement. ☐ ☐

Example 4.22. Consider the symmetric space of type AIII(2, $m > 4$). Take $\mu \in -\mathfrak{a}_s^+$ an indivisible one parameter subgroup. Then there are three possibilities for $I(\mu)$: we have $I(\bar{\alpha}^\vee_{1,m}) = S \setminus \{\alpha_{1,2}, \alpha_{m-1,m}\}$, $I(\bar{\alpha}^\vee_{1,m-1}) = S \setminus \{\alpha_{2,3}, \alpha_{m-2,m-1}\}$, and $I(\mu) = S^\sigma$ in the other cases. In the first situation, $[\mathfrak{l}_{I(\mu)}, \mathfrak{l}_{I(\mu)}]$ is isomorphic to the simple Lie algebra $\mathfrak{sl}_{m-2}$ and the induced involution is still of type AIII, but now of rank one. In the second situation, $[\mathfrak{l}_{I(\mu)}, \mathfrak{l}_{I(\mu)}]$ splits as a direct sum of three summands, one fixed by σ and the other two, each isomorphic to $\mathfrak{sl}_2$, exchanged by σ. Finally, in the third scenario, the isotropy group is in fact horospherical.

Remark 4.23. It is in fact very general that if the ray is in the interior of the valuation cone, then the closed orbit in the corresponding elementary embedding is horospherical, with an explicit description of its Lie algebra [BP87, Proposition 3.10].

Remark 4.24. Gagliardi and Hofscheier [GH15] obtained a description of the combinatorial data associated to any orbit in a spherical variety. We could in principle (neglecting the difficulty of describing the color map in general) have used this to show that facets of horosymmetric varieties are horosymmetric. However their result identifies only the conjugacy class of the isotropy subgroup, while we will need the precise knowledge of the isotropy group (actually Lie algebra) of a specific point in the orbit. On the other hand, it is possible to identify precisely the isotropy group of the point we consider, and not only its Lie algebra, by combining our result with that of [GH15]. This could be used to fully recover the description of orbit closures in wonderful compactifications given by De Concini and Procesi. We mention here, as it could be useful for other applications in Kähler geometry, that the work of Gagliardi and Hofscheier [GH15] further allows to identify the colored fan of a facet of a symmetric variety.

5. Linearized line bundles on horosymmetric varieties

In this section, we consider G-linearized line bundles on a G-horosymmetric variety X. We explain how to associate to such a line bundle $\mathcal{L}$ a priviledged B-invariant $\mathbb{Q}$-divisor, and several convex polytopes. For example, one can associate to $\mathcal{L}$ its (algebraic) moment polytope, and in the case X is toroidal, the moment polytope of the restriction of $\mathcal{L}$ to the toric subvariety. We determine the relations between these polytopes, and illustrate these notions on some examples, including the anticanonical line bundle.

Note that if G is simply connected, all line bundles on X are G-linearized [KKV89, KKLV89]. Else, if $\mathcal{L}$ is not linearized, there exists a tensor power $\mathcal{L}^k$ which is linearized. Finally, recall that any two linearizations of the same line bundle differ by a character of G.

5.1. **Special function associated to a linearized line bundle.** Let X be a horosymmetric embedding of G/H. Let $\mathcal{L}$ be a G-linearized line bundle on X. The action of G on $\mathcal{L}$ induces an action of G on meromorphic sections of $\mathcal{L}$, given explicitly by $(g\cdot s)(x) = g\cdot (s(g^{-1}\cdot x))$. Such a section is called B-semi-invariant if it is an eigenvector for the action of B, that is, there exists a character λ of B such that $b\cdot s = \lambda(b)s$.

A meromorphic section s of $\mathcal{L}$ defines a Cartier divisor $D_s = \mathrm{div}(s)$ representing $\mathcal{L}$. If s is B-semi-invariant, then d_s is B-invariant. There are two types of irreducible B-stable Weil divisor on X: the closure of colors $D\in\mathcal{D}$ of G/H, and the facets of X. Denote by $\mathcal{I}^G(X)$ the set of facets of X. Since D_s is by definition Cartier, it writes, by [Bri89, Proposition 3.1],

$$D_s = \sum_{Y\in\mathcal{I}^G(X)} v_s(\mu_Y)Y + \sum_{D\in\mathcal{D}_X} v_s(\rho(D))\overline{D} + \sum_{D\in\mathcal{D}\setminus\mathcal{D}_X} n_D\overline{D}$$

for some integers n_D and a piecewise linear integral function v_s on the fan $\mathcal{F}_X$.

In general there may not be any priviledged B-semi-invariant meromorphic section of $\mathcal{L}$. Given such a section s, with associated weight $\lambda_s \in \mathfrak{X}(B)$, the others are obtained by multiplying by eigenvectors f for the action of B on $\mathbb{C}(G/H)$, with eigenvalue an element λ_f of $\mathcal{M}$. Assume however that $\lambda_s|_{T_s} \in \mathfrak{X}(T_s)$ is induced by an element of $\mathcal{M}$ under the epimorphism $\mathfrak{X}(T_s) \to \mathcal{M} = \mathfrak{X}(T/T \cap H)$. Then we can choose f so that $(\lambda_s\lambda_f)|_{T_s}$ is trivial.

Definition 5.1. Assume there exists a section $s_{\mathcal{L}}$ such that $\lambda_s|_{T_s}$ is trivial. Then we call this section, well defined up to multiplicative scalar, the *special section* of $\mathcal{L}$, and $D_{\mathcal{L}} = \mathrm{div}(s_{\mathcal{L}})$ the *special divisor* representing $\mathcal{L}$.

In the general case, we may still define $D_{\mathcal{L}}$ as a $\mathbb{Q}$-divisor. Indeed, let s be a B-semi-invariant section of $\mathcal{L}$ with weight λ. There always exists a tensor power $\mathcal{L}^k$ of $\mathcal{L}$ such that the corresponding multisection $s^{\otimes k}$ has a B-weight λ whose restriction to T_s is induced by an element of $\mathcal{M}$. Thus the previous paragraph defines the special divisor $D_{\mathcal{L}^k}$.

Definition 5.2. In the situation described above, the *special divisor* of $\mathcal{L}$ is the $\mathbb{Q}$-divisor $D_{\mathcal{L}} = D_{\mathcal{L}^k}/k$.

Using the fact that $D_{\mathcal{L}\otimes\mathcal{L}}$ is Cartier, we may write

$$D_{\mathcal{L}} = \sum_{Y\in\mathcal{I}^G(X)} v_{\mathcal{L}}(\mu_Y)Y + \sum_{D\in\mathcal{D}_X} v_{\mathcal{L}}(\rho(D))\overline{D} + \sum_{D\in\mathcal{D}\setminus\mathcal{D}_X} n_{\mathcal{L},D}\overline{D}.$$

for some piecewise rational linear function $v_{\mathcal{L}}$ on $\mathcal{F}_X$.

Definition 5.3. The function $v_{\mathcal{L}}$ will be referred to as the *special function* associated to $\mathcal{L}$.

Remark that it takes non integral values if $\mathcal{L}$ admits no special section. Note that all of these notions are relative to the choice of a Borel subgroup B.

Definition 5.4. The unique $\bar{W}$-invariant function $v^t_{\mathcal{L}}$ defined by $v^t_{\mathcal{L}} = v_{\mathcal{L}}$ on $-\mathfrak{a}_s^+$, is the *toric special function* of $\mathfrak{L}$.

5.2. **Toroidal case.** When X is toroidal, we may identify the restriction of a G-linearized line bundle to the toric subvariety Z in terms of the objects defined in Section 5.1.

Proposition 5.5. *Let $\mathcal{L}$ be a G-linearized line bundle on a toroidal horosymmetric variety X. For a divisible enough integer m, the restriction $\mathcal{L}^m|_Z$ defines a $T/T\cap H \rtimes \bar{W}$-linearized toric line bundle on Z, such that the divisor associated with the $T/T\cap H \rtimes \bar{W}$-invariant section coincides with*

$$\sum_{F\in\mathcal{I}^{T/T\cap H}(Z)} m v^t_{\mathcal{L}}(u_F)F.$$

Proof. The restriction $\mathcal{L}|_Z$ inherits a linearization of the action of T_s as well as of the action of $N_K(T_s)\cap H$. The line bundle $\mathcal{L}^k$ for any k divisible enough admits a special section s. By definition of a special section, s is in particular $T_s\cap H$-invariant, hence $T_s\cap H$ acts trivially on $\mathcal{L}|_Z$. Thus the T_s-linearization of $\mathcal{L}^k$ actually comes from a $T/T\cap H$-linearization. The section $s|_Z$ is obviously $T/T\cap H$-invariant.

Let n_w be a representant in $N_K(T_s)\cap H$ of $w\in\bar{W}$. Then for any $t\in T_s$

$$n_w\cdot s(n_w^{-1}\cdot t\cdot x) = n_w\cdot s(n_w^{-1}tn_w\cdot x)$$

since $n_w \in H$,

$$= n_w n_w^{-1} t n_w \cdot s(x)$$

since $n \in N_K(T_s)$ and s is T_s-invariant,

$$= \chi(n_w) t \cdot s(x)$$

where χ is the character of H associated to $\mathcal{L}^{2k}$

$$= \chi(n_w) s(t \cdot x).$$

Since there is a finite number of n_w and they are in K, we may choose k so that $\chi(n_w) = 1$ for all $w \in \bar{W}$.

We now use the local structure of spherical varieties [BP87, Proposition 3.4]. Consider $\Delta = \bigcup_{D \in \mathcal{D}} D \subset G/H$ and set $U = X \setminus \bar{\Delta}$ and $V = Z \cap U$. Then V is the toric subvariety associated to the subfan contained in $-\mathfrak{a}_s^+$, and the toric divisors in V are precisely the $Y \cap V$ for $Y \in \mathcal{I}_X^G$. By [Bri89, Section 3.2], the restriction of $d_{\mathcal{L}^k}$ to V is

$$\operatorname{div}(s) \cap V = \sum_{Y \in \mathcal{I}^G(X)} k v_{\mathcal{L}}(\mu_Y)(Y \cap V).$$

By n_w-invariance of s, we obtain

$$\operatorname{div}(s) \cap w \cdot V = \sum_{Y \in \mathcal{I}^G(X)} k v_{\mathcal{L}}(\mu_Y) w \cdot (Y \cap V).$$

We deduce that

$$\operatorname{div}(s|_Z) = \sum_{F \in \mathcal{I}^{T/T \cap H}(Z)} k v_{\mathcal{L}}^t(u_F) F.$$

Remark that our reasoning with the representants n_w did not endow $\mathcal{L}^k|_Z$ with a $\bar{W}$-linearization *a priori*. However, $s|_Z$ is the (up to multiplicative scalar) $T/T \cap H$-invariant section of $\mathcal{L}^k|_Z$ and $\operatorname{div}(s|_Z)$ is $\bar{W}$-invariant, hence $\mathcal{L}^k|_Z$ admits a natural $\bar{W}$-linearization such that $w \cdot s|_Z = \mu(w) s|_Z$ for all $w \in \bar{W}$ and some character $\mu : \bar{W} \to \mathbb{S}^1$ of $\bar{W}$. The group $\bar{W}$ being finite, we may take a multiple m of k such that the character $\frac{m}{k}\mu$ is trivial, and obtain that the $T/T \cap H$-invariant section is also $\bar{W}$-invariant. □ □

5.3. **Polytopes.** To a G-linearized line bundle $\mathcal{L}$ on a complete horosymmetric variety X, we may associate several different convex polytopes. The first one is obtained directly from the special divisor of $\mathcal{L}$.

Definition 5.6. The *special polytope* $\Delta_{\mathcal{L}}$ of $\mathcal{L}$ is the convex polytope in $\mathcal{M} \otimes \mathbb{R}$ defined by the inequalities $m + v_{\mathcal{L}} \geq 0$, and $m(\rho(D)) + n_{D,\mathcal{L}} \geq 0$ for all $D \in \mathcal{D} \setminus \mathcal{D}_X$.

Definition 5.7. The *toric polytope* $\Delta_{\mathcal{L}}$ of $\mathcal{L}$ is the convex polytope defined by

$$\Delta_{\mathcal{L}}^t = \{m \in \mathcal{M} \otimes \mathbb{R}; m + v_{\mathcal{L}}^t \geq 0\}.$$

Remark that the toric polytope is $\bar{W}$-invariant (and independent of the choice of a Borel subgroup B containing T).

Definition 5.8. The *moment polytope* $\Delta_{\mathcal{L}}^+$ is the set defined as the closure in $\mathfrak{X}(T) \otimes \mathbb{R}$ of the set of all λ/k such that there exists a non-zero B-semi-invariant global holomorphic section s of $\mathcal{L}^k$ with weight λ (that is, $b \cdot s = \lambda(b) s$ for all $b \in B$).

Note that all of these sets are multiplicative with respect to tensor powers, that is $\Delta^{\heartsuit}_{\mathcal{L}^m} = m\Delta^{\heartsuit}_{\mathcal{L}}$ for any positive integer m.

It was proved by Brion that the moment polytope is indeed a convex polytope. More precisely, we have the following relation between the special polytope and the moment polytope.

Proposition 5.9. *Let $\chi \in \mathfrak{X}(H)$ be the character associated to the restriction of $\mathcal{L}$ to G/H. Consider χ as before as an element of $\mathfrak{X}(T/T_s) \otimes \mathbb{R} \subset \mathfrak{X}(T) \otimes \mathbb{R}$ via its restriction to $T \cap H$. Then*

$$\Delta^+_{\mathcal{L}} = \chi + \Delta_{\mathcal{L}}.$$

Proof. By multiplicativity, we may as well prove the result for $\mathcal{L}^k$. This has two consequences: we may choose k so that $\mathcal{L}$ has a special section, and $k\chi|T_s \cap H = 0$.

Let s denote the special section of $\mathcal{L}^k$, and denote by $\lambda \in \mathfrak{X}(B) = \mathfrak{X}(T)$ its character. By [Bri89, Proposition 3.3] (see also [Bri, Section 5.3]), we have

$$\Delta^+_{\mathcal{L}^k} = \lambda + \Delta_{\mathcal{L}^k}.$$

We know that $\lambda|_{T_s} = 0$ hence we may consider λ as an element of $\mathfrak{X}(T/T_s) \subset \mathfrak{X}(T)$.

From the other consequence of considering $\mathcal{L}^k$, we see that we may also consider $k\chi$ as an element of $\mathfrak{X}(T \cap H/T_s \cap H)$. The natural epimorphism $T \cap H \to T/T_s$ identifies $T \cap H/T_s \cap H$ with T/T_s. Let $t \in T \cap H$. We have, by definition of s,

$$\begin{aligned} \lambda(t)s(eH) &= t \cdot s(t^{-1}H) \\ &= t \cdot s(eH) \end{aligned}$$

since $t \in H$

$$= (k\chi)(t)s(eH)$$

by definition of χ, hence the theorem. □ □

Recall the caracterization of ample and globally generated line bundles proved by Brion. Given a maximal cone $\mathcal{C}$ contained in $\mathcal{F}_X$, let $m_{\mathcal{C}}$ denote the element of $\mathcal{M} \otimes \mathbb{Q}$ such that $v_{\mathcal{L}}(y) = m_{\mathcal{C}}(y)$ for $y \in \mathcal{C}$.

Proposition 5.10 ([Bri89, Théorème 3.3])**.** *The G-linearized line bundle $\mathcal{L}$ is* globally generated *if and only if*

- *The function $v_{\mathcal{L}}$ is convex and*
- *$n_{D,\mathcal{L}} \geq m_{\mathcal{C}}(\rho(D))$ for all $D \in \mathcal{D} \setminus \mathcal{D}_X$ and maximal cone $\mathcal{C} \in \mathcal{F}_X$.*

It is ample *if and only if it is globally generated and furthermore*

- *$m_{\mathcal{C}_1} \neq m_{\mathcal{C}_2}$ if $\mathcal{C}_1 \neq \mathcal{C}_2 \in \mathcal{F}_X$ are two maximal cones,*
- *$n_{D,\mathcal{L}} \neq m_{\mathcal{C}}(\rho(D))$ for all $D \in \mathcal{D} \setminus \mathcal{D}_X$ and maximal cone $\mathcal{C} \in \mathcal{F}_X$.*

Definition 5.11. The *support function* $w_\Delta : V^* \longrightarrow \mathbb{R}$ of a convex polytope Δ in a real vector space V is defined by

$$w_\Delta(x) = \sup\{m(x); m \in \Delta\}.$$

One may recover the convex polytope Δ from its support function by checking $\Delta = \{m \in V; m \leq w_\Delta\}$. As a consequence from this definition, we have:

Corollary 5.12. *If $\mathcal{L}$ is globally generated, then $v_{\mathcal{L}}(y) = w_{\Delta_{\mathcal{L}}}(-y)$ for $y \in |\mathcal{F}_X|$.*

Let C^+ denote the positive Weyl chamber in $\mathfrak{a}^* = \mathfrak{X}(T) \otimes \mathbb{R}$, which may be defined as

$$C^+ = \{p \in \mathfrak{a}^*; p(\alpha^\vee) \geq 0, \forall \alpha \in \Phi^+\}.$$

Similarly, we define the positive restricted Weyl chamber in $\mathfrak{a}_s^*$ as

$$\bar{C}^+ = \{p \in \mathfrak{a}_s^*; p(\bar{\alpha}^\vee) \geq 0, \forall \bar{\alpha} \in \bar{\Phi}^+\}.$$

Proposition 5.13. *The polytope $\Delta_{\mathcal{L}}$ is a translate by an element of $\bar{C}^+$ of a polytope which is the intersection of a $\bar{W}$-invariant polytope with $\bar{C}^+$. In particular, $\Delta_{\mathcal{L}} \subset \bar{C}^+$ and*

$$\Delta_{\mathcal{L}}^t = \mathrm{Conv}(\bar{W} \cdot \Delta_{\mathcal{L}}).$$

Proof. By definition, the polytope $\Delta_{\mathcal{L}}$ has outer normal along codimension one faces which are given by some elements of the valuation cone and the images by ρ of some colors. Since the only images of colors that are not in the valuation cone are simple restricted coroots by Proposition 2.24 we obtain at once that $\Delta_{\mathcal{L}}$ is a translate of a polytope which is the intersection of a $\bar{W}$-invariant polytope with $\bar{C}^+$.

To check that it is a translation by an element of $\bar{C}^+$, it is enough to check that $\Delta_{\mathcal{L}}$ is included in $\bar{C}^+$ itself. This is a direct consequence of the relation $\Delta_{\mathcal{L}}^+ = \chi + \Delta_{\mathcal{L}}$ together with the fact that $\Delta_{\mathcal{L}}^+ \in C^+$ by definition. Indeed, given $p \in \Delta_{\mathcal{L}}$ and $\alpha \in \Phi_s^+$, we have

$$p(\bar{\alpha}^\vee) = (p + \chi)(\bar{\alpha}^\vee) - \chi(\bar{\alpha}^\vee)$$

which is positive since χ is zero on $\mathfrak{a}_s$, $p+\chi \in \Delta_{\mathcal{L}}^+ \subset C^+$ and $\bar{\alpha}^\vee$ is a positive multiple of either a positive coroot or the sum of two positive coroots by Definition 2.22.
□ □

Recall that the linear part of a cone containing the origin is the largest linear subspace included in the cone.

Corollary 5.14. *The following conditions are equivalent:*

(1) *$\mathcal{L}^m$ admits a global holomorphic Q-semi-invariant section for some $m \in \mathbb{N}^*$,*
(2) *$\Delta_{\mathcal{L}}^+ \cap \mathfrak{X}(T/T \cap [L, L]) \otimes \mathbb{R} \neq \emptyset$*
(3) *$\Delta_{\mathcal{L}}$ intersects the linear part of $\bar{C}^+$,*
(4) *$\Delta_{\mathcal{L}}^t \cap \bar{C}^+ = \Delta_{\mathcal{L}} = -\chi + \Delta_{\mathcal{L}}^+$.*

Proof. The first condition translates directly into a condition on $\Delta_{\mathcal{L}}^+$: it is equivalent to the fact that some $\mathcal{L}^m$ admits a global holomorphic B-semi-invariant section whose weight is in $\mathfrak{X}(T/T \cap [L, L])$, that is, $\Delta_{\mathcal{L}}^+ \cap \mathfrak{X}(T/T \cap [L, L]) \neq \emptyset$.

One checks easily that the linear part of $\bar{C}^+ \subset \mathfrak{X}(T_s) \otimes \mathbb{R}$ is $\mathfrak{X}(T_s/T_s \cap [L, L]) \otimes \mathbb{R}$, and coincides also with the linear subspace of $\bar{W}$-invariant elements of $\mathfrak{X}(T_s) \otimes \mathbb{R}$. Now since $\Delta_{\mathcal{L}}^+ = \Delta_{\mathcal{L}} + \chi$ and $\chi \in \mathfrak{X}(T/(([L, L] \cap T)T_s)) \otimes \mathbb{R} \subset \mathfrak{X}(T/[L, L] \cap T) \otimes \mathbb{R}$, the first condition is equivalent to $\Delta_{\mathcal{L}} \cap \mathfrak{X}(T_s/T_s \cap [L, L]) \otimes \mathbb{R} \neq \emptyset$.

Finally, thanks to Proposition 5.13, we obtain the equivalence with the last condition. □ □

5.4. **The anticanonical line bundle.** Recall the Weil divisor representing the anticanonical class obtained by Brion on any spherical variety.

Proposition 5.15 ([Bri97, Sections 4.1 and 4.2]). *The horosymmetric variety X admits an anticanonical Weil divisor*

$$-K_X = \sum_{Y\in\mathcal{I}^G(X)} Y + \sum_{D\in\mathcal{D}} m_D\overline{D}$$

where the m_j are positive integers with an explicit description in terms of the colored data.

More precisely, if one considers the subvariety $\hat{X}\subset X$ which consists of all the orbits of codimension strictly less than two in X, it is a smooth variety with a well defined anticanonical line bundle, and this anticanonical line bundle admits a B-semi-invariant section s with weight λ whose divisor is the divisor $-K_X$ above. In our horosymmetric situation, the weight λ is equal to $\sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha$.

We may reason as if $\lambda|_{T_s} = \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha\circ\mathcal{P}$ is induced by an element of $\mathfrak{X}(T/T\cap H)$, up to passing to $K_{\hat{X}}^{-k}$ for some positive integer k if necessary. Let $h\in\mathbb{C}(\hat{X})$ be a B-semi-invariant function with weight $-\sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha\circ\mathcal{P}$. Then hs is the special section of $K_{\hat{X}}^{-1}$. Its B-weight is $\sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha\circ\mathcal{H}$. Note that this is equal to $\sum_{\alpha\in\Phi_{Q^u}}\alpha\circ\mathcal{H}$ since $\sum_{\alpha\in\Phi_s^+}\alpha\circ\mathcal{P}=\sum_{\alpha\in\Phi_s^+}\alpha$. The special divisor D^{ac} of $K_{\hat{X}}^{-1}$ is thus

$$\begin{aligned} D^{ac} = &\sum_{Y\in\mathcal{I}_X^G}\left(1-\sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha\circ\mathcal{P}(\mu_Y)\right)Y \\ &+\sum_{D\in\mathcal{D}}\left(m_D-\sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha\circ\mathcal{P}(\rho(D))\right)\overline{D}. \end{aligned}$$

Note that $\hat{X}$ is a global parabolic induction with respect to the morphism $f:\hat{X}\to G/P$ extending the natural morphism $G/H\to G/P$. We accordingly decompose the anticanonical line bundle as $K_{\hat{X}}^{-1}=K_f^{-1}\otimes f^*K_{G/P}^{-1}$. By definition, the special section is the product of special sections of these naturally linearized line bundles and the divisor D^{ac} on $\hat{X}$ is the sum of their respective divisors D_f^{ac} and D_P^{ac}. The special section of $f^*K_{G/P}^{-1}$ is obviously Q-semi-invariant, with weight precisely equal to $\sum_{\alpha\in\Phi_{Q^u}}\alpha\circ\mathcal{H}$. As a consequence, the special section of K_f^{-1} is G-invariant. The special section of $K_{\hat{X}}^{-1}$ is thus Q-semi-invariant. It follows from this discussion that the coefficients of colors coming from $L/L\cap H$ must vanish.

By normality of X, the special divisors D_f^{ac} of K_f^{-1} and D_P^{ac} of $f^*K_{G/P}^{-1}$, defined on $\hat{X}$, extend to X as Weil divisors and $D^{ac}=D_f^{ac}+D_P^{ac}$ holds on X. We further have explicitly

$$D_f^{ac}=\sum_{Y\in\mathcal{I}_X^G}\left(1-\sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\alpha\circ\mathcal{P}(\mu_Y)\right)Y$$

and

$$D_P^{ac}=\sum_{\alpha\in\Phi_{Q^u}\cap S}\left(m_{D_\alpha}-\sum_{\beta\in\Phi_{Q^u}\cup\Phi_s^+}\beta\circ\mathcal{P}(\rho(D_\alpha))\right)\overline{D_\alpha}.$$

5.5. **Examples.**

Example 5.16. We consider again the variety $X = \mathbb{P}^1 \times \mathbb{P}^1$ equipped with the diagonal action of SL_2. The line bundles on X are the $\mathcal{O}(k,m)$ for $k,m \in \mathbb{Z}$. They admit natural $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearization hence also a natural linearization under the diagonal action. There are two colors D^+ and D^- with same image $\bar{\alpha_{1,2}}^\vee$ via the color map, the fan of X is the negative Weyl chamber, a single ray generated by $-\bar{\alpha}_{1,2}^\vee$ corresponding to the orbit $Y = \mathrm{diag}(\mathbb{P}^1)$. The line bundle corresponding to D^+ is, say, $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ corresponds to D^-, while the line bundle correponding to Y is obviously $\mathcal{O}(1,1)$.

The special divisor corresponding to $\mathcal{O}(k,m)$ is $\frac{k+m}{2}Y + \frac{k-m}{2}(D^+ - D^-)$, its moment polytope is the set of all $t\alpha_{1,2}$ for $\frac{|k-m|}{2} \leq t \leq \frac{k+m}{2}$. It is the same as the special polytope. The toric polytope is the set of all $t\alpha_{1,2}$ for $|t| \leq \frac{k+m}{2}$.

Example 5.17. Consider the wonderful compactification X of Type AIII(2, $m > 4$), under the action of SL_m. Since this group is simple and simply connected, all line bundles admit a unique linearization. It follows from [Bri89] that the Picard group of X is the free abelian group generated by the three colors D_1^+, D_1^- and D_2 whose images under the color map are $\rho(D_1^{\pm}) = \bar{\alpha}_{2,m-1}^\vee$ and $\rho(D_2) = \bar{\alpha}_{1,2}^\vee$. The two G-invariant prime divisors Y_1, corresponding to the ray generated by $\mu_1 = -\bar{\alpha}_{1,m-1}^\vee$ and Y_2, corresponding to the ray generated by $\mu_2 = -\bar{\alpha}_{1,m}^\vee$ write in this basis as $Y_1 = D_1^+ + D_1^- - D_2$ and $Y_2 = 2D_2 - D_1^+ - D_1^-$. Given a line bundle corresponding to the divisor $k_1^+ D_1^+ + k_1^- D_1^- + k_2 D_2$, the corresponding special divisor is $b_1 Y_1 + b_2 Y_2 + b^{\pm}(D_1^+ + D_1^-)$ where $b_1 = k_1^+ + k_1^- + k_2$, $b_2 = \frac{k_1^+ + k_1^-}{2} + k_2$ and $b^{\pm} = \frac{k_1^+ - k_1^-}{2}$.

Assume $b^{\pm} = 0$. The polytope Δ^t is the convex hull of the images by $\bar{W}$ of the point $b_2\bar{\alpha}_{1,2} + b_1\bar{\alpha}_{2,3}$, provided it is in the positive restricted Weyl chamber, that is $2b_2 \geq b_1$ and $b_1 \geq b_2$. Note that Brion's ampleness criterion translates here as the fact that this point is in the interior of the positive restricted Weyl chamber. The polytope $\Delta^+ = \Delta$ is the intersection of Δ^t with the positive chamber. If $b^{\pm}$ is non-zero, then we must intersect with another half-plane to get the polytope.

6. Metrics on line bundles

We will now use the objects introduced in the previous section to study hermitian metrics on G-linearized line bundles. Given a hermitian metric q on $\mathcal{L}$, recall that its local potentials are the functions $\psi : y \mapsto -\ln|s(y)|^2$ where s is a local trivialization of $\mathcal{L}$. We allow for now singular hermitian metrics, that is the local potentials are only required to be locally integrable. The metric is called locally bounded/continuous/smooth if and only if its local potentials are. It is called non-negatively curved (in the sense of currents) if the local potentials are plurisubharmonic functions and positively curved if its local potentials are strictly plurisubharmonic.

6.1. **Asymptotic behavior of toric potentials.**

Proposition 6.1. *Let $G/H \subset X$ be a complete horosymmetric variety, and $\mathcal{L}$ a G-linearized line bundle on X, with special function $v_{\mathcal{L}}$. Let q be a K-invariant locally bounded metric on $\mathcal{L}$ with toric potential u. Then the function*

$$x \mapsto u(x) - 2v_{\mathcal{L}}^t(x)$$

is bounded on $\mathfrak{a}_s$*.*

The proof will use the process of discoloration, which allows to reduce to the case of a toroidal variety.

Definition 6.2. Let (X, x) be an embedding of G/H with colored fan $\mathcal{F}_X$. Then the *discoloration* (X', x') of (X, x) is the embedding of G/H whose colored fan $\mathcal{F}_{X'}$ is obtained by taking the collection of all colored cones of the form $(\mathcal{C} \cap \mathcal{V}, \emptyset)$ for $(\mathcal{C}, \mathcal{R}) \in \mathcal{F}_X$ and their faces.

The discoloration (X', x') of (X, x) is equipped with a G-equivariant birational proper morphism $d' : X' \to X$ sending x' to x. The simplest example of discoloration is given by the blow up of $\mathbb{P}^2$ at 0, seen as a horospherical variety under the action of SL_2.

Proof. Note that the toric potential is defined only up to an additive scalar, but this does not affect the statement. The choice of toric potential is determined by the choice of a non zero element $\xi \in \mathcal{L}_{eH}$. We fix such a choice here.

Since the special function of $\mathcal{L}^m$ is $mv_{\mathcal{L}}$ and the toric potential of $q^{\otimes m}$ is mu, replacing $\mathcal{L}$ by a power of $\mathcal{L}$ will not affect the result. For example we can already assume that $\mathcal{L}$ admits a special section.

Consider the pullback $\mathcal{L}'$ of $\mathcal{L}$ under the discoloration morphism $d' : X' \to X$, equipped with the metric d'^*q. The special function $v_{\mathcal{L}'}$ coincides with the special function $v_{\mathcal{L}}$ [Pas17, Proof of Lemma 5.3]. Furthermore, by Proposition 5.5 and up to replacing $\mathcal{L}$ by a power of itself, the restriction of $\mathcal{L}'$ to the toric subvariety $Z' \subset X'$ is a $T/T \cap H$-linearized line bundle with divisor

$$\sum_{F \in \mathcal{I}^{T/T\cap H}(Z')} v_{\mathcal{L}}^t(u_F) F.$$

Consider the Batyrev-Tschinkel metric associated to this line bundle [Mai00, Section 3.3]. It is a compact torus invariant, continuous metric on $\mathcal{L}'|_{Z'}$ with toric potential $u_{BT} : x \in\mapsto -2\ln|\exp_s(x) \cdot \xi|_{BT}$ equal to

$$x \mapsto 2v_{\mathcal{L}}^t(x).$$

Beware that here $\exp_s$ denotes the exponential map for the Lie group $T/T \cap H$, which does not coincide with the exponential map for G. Here however, since the $T/T \cap H$-linearization of $\mathcal{L}'|_{Z'}$ was obtained *via* factorization of the T_s-linearization, we have $\exp_s(x) \cdot \xi = \exp(x) \cdot \xi$. We then have

$$(u - u_{BT})(x) = -2\ln \frac{|\exp(x) \cdot \xi|_{BT}}{|\exp(x) \cdot \xi|_q}.$$

Since the Batyrev-Tschinkel metric is continuous and the metric q is locally bounded, we obtain that the above function is globally bounded, hence the statement. □ □

6.2. **Positive metrics on globally generated line bundles.** In this section, X is a horosymmetric variety and $\mathcal{L}$ is a globally generated and big line bundle on X.

Proposition 6.3. *Let q be a non-negatively curved, K-invariant, locally bounded hermitian metric on $\mathcal{L}$ with toric potential u. Assume in addition that q is locally bounded, and that its restriction to $\mathcal{L}|_{G/H}$ is smooth and positively curved. Then*

1. *u is a smooth, strictly convex, $\bar{W}$-invariant function,*
2. *there exists a constant C such that $w_{-2\Delta^t} - C \leq u \leq w_{-2\Delta^t} + u(0)$,*

(3) $a \mapsto d_a u$ *defines a diffeomorphism from* $\mathfrak{a}_s$ *onto* $\mathrm{Int}(-2\Delta^t)$,
(4) $a \mapsto d_a u$ *defines a diffeomorphism from* $\mathrm{Int}(\mathfrak{a}_s^+)$ *onto* $\mathrm{Int}(-2\Delta^t \cap \bar{C}^+)$.

Proof. Without loss of generality, we may assume that X is toroidal *via* the discoloration procedure. Then the first property directly follows from restriction to the toric subvariety. The second property is a translation of Proposition 6.1, with the additional input that by convexity and since $w_{-2\Delta^t}$ is piecewise linear, we can take $u(0)$ as constant on one side. Then the third property is a consequence of the second, and the fourth follows by $\bar{W}$-invariance. □ □

Remark 6.4. Note that the open dense orbit is contained in the ample locus of any big line bundle on X, hence there are hermitian metrics as in the statement of Proposition 6.3.

The *convex conjugate* $u^* : \mathfrak{a}_s^* \to \mathbb{R} \cup \{+\infty\}$ of u is the convex function defined by

$$u^*(p) = \sup_{y \in \mathfrak{a}_s} (p(y) - u(y)).$$

If u is the toric potential of a metric q as in Proposition 6.3, then u^* is W-invariant and $u^* = +\infty$ on $\mathfrak{a}_s^* \setminus -2\Delta^t$. Furthermore, we have $u^*(p) = p(a) - u(a)$ whenever $p = d_a u \in \mathrm{Int}(-2\Delta^t)$.

6.3. **Metric induced on a facet.** Let $\mathcal{L}$ be a G-linearized line bundle on a horosymmetric embedding (X, x) of G/H. Assume that $\mathcal{L}$ admits a special section s and write the special divisor as $D_{\mathcal{L}} = \sum_Y n_y Y + \sum_D n_D \overline{D}$. For every facet Y of X, let $\mu_Y \in \mathfrak{Y}(T_s)$ denote the indivisible generator of the ray corresponding to Y in the colored fan of X, denote by $E_Y \subset X$ the corresponding elementary embedding and let $x_Y = \lim \mu_Y(z) \cdot x$.

For each facet Y, we choose a complement $\mathfrak{a}_Y$ of $\mathbb{R}\mu_Y$ in $\mathfrak{a}_s$ as in Section 4.3, corresponding to a torus $T_{s,Y}$. Note that the torus $T_{s,Y}$ is a maximal split torus for the involution associated to Y as in Section 4.3.

Let h be a hermitian metric on $\mathcal{L}$ and assume that it is smooth on the elementary embedding E_Y. Denote by $\psi : b \mapsto -2 \ln |s(b)|_h$ the potential of h with respect to the special section s.

There exists a unique $\lambda \in \mathfrak{X}(T/(T \cap H)T_Y) \otimes \mathbb{Q}$ such that $\lambda(\mu_Y) = -n_Y$. Up to taking a tensor power of $\mathcal{L}$, we may thus find a rational function $f \in \mathbb{C}(X)$ such that $\mathrm{ord}_Y(f) = -n_Y$ and $f(x) = 1$. Let $s_\lambda = fs$ denote a B-semi-invariant section obtained by multiplying the section s by f. Then the section s_λ does not vanish identically on Y, and its restriction to Y is further a special section for $\mathcal{L}|_Y$. The potential ψ_λ of h with respect to s_λ is defined on the whole E_Y and satisfies, for $b \in B$, $\psi_\lambda(bH) = \psi(bH) - 2 \ln \lambda(b)$.

The toric potential of h is $u(a) = \psi(\exp(a) \cdot x)$ and the toric potential of the restriction of h to $\mathcal{L}|_Y$ is the function u_Y defined by $u_Y(b) = \psi_\lambda(\exp(b) \cdot x_Y)$ for $b \in \mathfrak{a}_Y$. Let us also define the function u_λ on $\mathfrak{a}_s$ by $u_\lambda(a) = \psi_\lambda(\exp(a) \cdot x) = u(a) - 2\lambda(a)$.

Proposition 6.5. *Let* (t_j), $(\tilde{t}_j)$ *be sequences of real numbers and* (b_j), $(\tilde{b}_j)$ *be sequences of elements of* $\mathfrak{a}_Y$ *such that* $\lim t_j = -\infty$, $(\tilde{t}_j e^{t_j})$ *is bounded,* $\lim b_j = b \in \mathfrak{a}_Y$ *and* $\lim \tilde{b}_j = \tilde{b} \in \mathfrak{a}_Y$*. Then*

$$\lim u_\lambda(t_j \mu_Y + b_j) = u_Y(b)$$

and

$$\lim d_{t_j\mu_Y+b_j}u_\lambda(\tilde{t}_j\mu_Y+\tilde{b}_j)=d_bu_Y(\tilde{b}).$$

Proof. The first statement follows directly from the smoothness of h, hence of ψ_λ.
For the second limit, the result holds because

$$\lim \tilde{t}_j d_{t_j\mu_Y+b_j}u_\lambda(\mu_Y)=0.$$

This is easily verified by noticing $\exp(-t)d_{t\mu_Y+b}u_\lambda(\mu_Y)=d_{t\mu_Y+b}\psi_\lambda(\partial/\partial t)$ where $\partial/\partial t$ is the constant real direction vector field of $\mathbb{C}$ identified with the $\mathbb{C}$-factor in the toric subvariety $Z\simeq\mathbb{C}\times T_Y$ in E_Y. □ □

Assume now that h is positively curved on the elementary embedding $E_Y\subset X$. Then we may consider the convex conjugates u^*, u^*_λ and u^*_Y.

Proposition 6.6. *Let $b\in\mathfrak{a}_Y$. Then at $p=d_bu_Y$ we have*

$$u^*_Y(p)=u^*_\lambda(p)=u^*(p+2\lambda).$$

Proof. The second inequality follows from elementary properties of the convex conjugate. For the other equality, we have

$$\begin{aligned}u^*_Y(d_bu_Y)&=d_bu_Y(b)-u_Y(b)\\&=\lim d_{t_j\mu_Y+b_j}u_\lambda(t_j\mu_Y+b_j)-u_\lambda(t_j\mu_Y+b_j)\end{aligned}$$

for any sequences such that $\lim t_j=-\infty$ and $\lim b_j=b$ by Proposition 6.5

$$=\lim u^*_\lambda(d_{t_j\mu_Y+b_j}u_\lambda).$$

Note that the convex conjugate u_{λ^*} is not *a priori* continuous up to the boundary, hence it is not enough to conclude. On the other hand, if we choose $t\in\mathbb{R}$ we have

$$u^*_\lambda(d_bu_Y)=\lim_{s\to1}u^*_\lambda(sd_bu_y+(1-s)d_{t\mu_Y+b}u_\lambda).$$

We may find t_s and b_s such that $sd_bu_y+(1-s)d_{t\mu_Y+b}u_\lambda=d_{t_s\mu_Y+b_s}u_\lambda$. The fact that $\lim_{s\to1}d_{t_s\mu_Y+b_s}u_\lambda=d_bu_Y$ ensures that $\lim t_s=-\infty$ and that $\lim b_s=b$ (it certainly ensures that b_s is bounded, then considering converging subsequences yields the limit since u_Y is smooth and strictly convex), hence the statement. □
□

It is clear from the point of view of the toric subvariety Z that the domain of u^*_Y is the toric polytope $-2\Delta^t_Y+2\lambda$ of the restriction of $\mathcal{L}$ to Y, translated, which is the facet of $-2\Delta^t+2\lambda$ whose outer normal is μ_Y. Concerning moment polytopes, we have:

Proposition 6.7 ([Bri])**.** *The moment polytope of Y is the codimension one face of Δ^+ whose outer normal in the affine space $\chi+\mathcal{M}_\mathbb{R}$ is $-\mu_Y$.*

The special polytope Δ_Y of $\mathcal{L}|_Y$ is then $\Delta^+_Y-\chi-\lambda$ and we have $\chi_Y=\chi+\lambda$ under the usual identifications.

6.4. **Volume of a polarized horosymmetric variety.** Before applying the results from this section combined with our computation of the Monge-Ampère operator to get an integration formula on horosymmetric varieties, we recall the formula for the volume of a horosymmetric variety, consequence of a general result of Brion.

Proposition 6.8 ([Bri89, Théorème 4.1])**.** *Let X be a projective horosymmetric variety, and $\mathcal{L}$ be a G-linearized ample line line bundle on X. Then*

$$\mathcal{L}^n = n! \int_{\Delta_{\mathcal{L}}^+} \prod_{\Phi^+ \setminus E} \frac{\kappa(\alpha, p)}{\kappa(\alpha, \rho)} dp$$

where E is the set of roots $\alpha \in \Phi^+$ that are orthogonal to $\Delta_{\mathcal{L}}^+$ with respect to κ and dp is the Lebesgue measure on the affine span of $\Delta_{\mathcal{L}}^+$, normalized by the translated lattice $\chi + \mathfrak{X}(T/T \cap H)$.

In the case of horosymmetric varieties, the set E is exactly $\Phi^+ \cap \Phi_L^\sigma$. We will use the notation

$$P_{DH}(p) = \prod_{\Phi_{Q^u} \cup \Phi_s^+} \frac{\kappa(\alpha, p)}{\kappa(\alpha, \rho)} = \prod_{\Phi_{Q^u} \cup \Phi_s^+} \frac{\kappa(\alpha, \alpha)}{2\kappa(\alpha, \rho)} p(\alpha^\vee)$$

6.5. **Integration on horosymmetric varieties.** Let $\mathcal{L}$ be a globally generated and big G-linearized line bundle on a horosymmetric variety X. We assume in this subsection that h is a locally bounded, non-negatively curved metric on $\mathcal{L}$, smooth and positive on the restriction of $\mathcal{L}$ to G/H. We denote by ω its curvature current. Assume furthermore that χ vanishes on $[\mathfrak{l}, \mathfrak{l}]$.

Let ψ denote a K-invariant function on X, integrable with respect to ω^n, and continuous on G/H. To simplify notations, we denote by $\psi(a)$ the image by ψ of $\exp(a)H$ for $a \in \mathfrak{a}_s$. Let Δ' denote the polytope $-2\Delta^t \cap \bar{C}^-$.

Proposition 6.9. *Let dq denote the Lebesgue measure on the affine span of Δ^+, normalized by the lattice $\chi + \mathcal{M}$, let dp denote the Lebesgue measure on $\mathcal{M} \otimes \mathbb{R}$ normalized by $\mathcal{M}$. Then there exist a constant C_H', independent of h and ψ, such that*

$$\begin{aligned}\int_X \psi \omega^n &= \frac{C_H'}{2^n} \int_{\Delta'} \psi(d_p u^*) P_{DH}(2\chi - p) dp \\ &= C_H' \int_{\chi + \Delta^t \cap \bar{C}^+} \psi(d_{2\chi - 2q} u^*) P_{DH}(q) dq.\end{aligned}$$

Proof. Since ω^n puts no mass on $X \setminus G/H$, we may first note that

$$\int_X \psi \omega^n = \int_{G/H} \psi \omega^n$$

Then by K-invariance and Proposition 3.7, this is equal to

$$C_H \int_{-\mathfrak{a}_s^+} \psi(a) J_H(a) \frac{\omega^n}{dV_H} (\exp(a)H) da.$$

Now by definition of J_H and Corollary 3.13, this is equal to

$$C_H \int_{-\mathfrak{a}_s^+} \frac{n!}{2^{2r + |\Phi_{Q^u}|}} \psi(a) \prod_{\alpha \in \Phi_{Q^u}} (2\chi - d_a u)(\alpha^\vee) \prod_{\beta \in \Phi_s^+} (-d_a u)(\beta^\vee) \det(d^2 u)(a) da$$

We then use the change of variables $2p = d_a u$ and thanks to Proposition 6.3 we obtain

$$\int_X \psi\omega^n = C_H \int_{(-\Delta^t)\cap\bar{C}^-} n!2^{|\Phi_s^+|-r}\psi(d_{2p}u^*) \prod_{\alpha\in\Phi_{Q^u}} (\chi-p)(\alpha^\vee) \prod_{\beta\in\Phi_s^+}(-p)(\beta^\vee)dp$$

where dp is for the moment a Lebesgue measure independent of ψ. Actually by considering a constant ψ, hence the volume of $\mathcal{L}$, we see that the Lebesgue measure is further independent of the choice of q.

The assumption that χ vanishes on $[\mathfrak{l},\mathfrak{l}]$ ensures that $\chi(\beta^\vee)=0$ for all $\beta\in\Phi_s^+$, hence using the change of variables $q=\chi-p$, we get

$$\int_X \psi\omega^n = n!2^{|\Phi_s^+|-r}C_H \int_{\chi-(-\Delta^t)\cap\bar{C}^-} \psi(d_{2\chi-2q}u^*) \prod_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} q(\alpha^\vee)dq.$$

Taking the constant C'_H to be the covolume of the lattice $\mathfrak{X}(T/T\cap H)$ under dp times the constant

$$n!2^{2|\Phi_s^+|+|\Phi_{Q^u}|-r}C_H \prod_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} \frac{\kappa(\alpha,\rho)}{\kappa(\alpha,\alpha)}$$

we obtain the result. □ □

Corollary 6.10. *Assume that $\mathcal{L}^m$ admits a global Q-semi-invariant section for some $m>0$. Then the constant C'_H in Proposition 6.9 is equal to $n!(2\pi)^n$ and the integration is over Δ^+ in the second equality.*

Proof. It follows from applying Proposition 6.9 to the constant function $\psi=\frac{1}{(2\pi)^n}$, using Corollary 5.14 to check that the integration is over Δ^+, and comparing with Brion's formula for the degree of a line bundle (Proposition 6.8). □ □

7. Mabuchi Functional and coercivity criterion

7.1. **Setting.** We fix in this section a $\mathbb{Q}$-line bundle $\mathcal{L}$ on a n-dimensional smooth horosymmetric variety X. We assume that there exists a positive integer m such that $\mathcal{L}^m$ is an ample line bundle, with a fixed G-linearization, and that it admits a global holomorphic Q-semi-invariant section. Let Δ^+ denote the moment polytope of $\mathcal{L}$, and $\Delta'=-2(\Delta^+-\chi)$ where χ is the isotropy character of $\mathcal{L}$. As a consequence of Corollary 5.14, we may fix a point λ_0 in the relative interior of $\Delta^+\cap\mathfrak{X}(T/T\cap [L,L])\otimes\mathbb{R}$. The point $2(\chi-\lambda_0)$ is then in the interior of $\Delta'\cap\mathfrak{X}(T_s/T_s\cap[L,L])\otimes\mathbb{R}$.

We will make the following additional assumptions:

(T) the horosymmetric variety X is toroidal, and whenever a facet of Δ^+ intersects a Weyl wall, either the facet is fully contained in the wall or its normal belongs to the wall, furthermore, Δ^+ intersects only walls defined by roots in Φ_L,

(R) for any restricted Weyl wall, there are at least two roots in Φ_s^+ that vanish on this Weyl wall.

Unlike the assumption that $\mathcal{L}$ is trivial on the symmetric fibers, these assumptions are very likely not meaningful. We use these to provide a rather general application of the setting we developped for Kähler geometry on horosymmetric varieties in a paper with reasonnable length. We have no claim of giving the most general statement, and expect that at least assumption (R) can be removed without too much difficulties. Once this is achieved, removing assumption (T) should require an analysis similar to that given by Li-Zhou-Zhu in [LZZ] to treat non-toroidal group compactifications. Finally, removing the assumption that $\mathcal{L}$ is trivial on the symmetric fibers appears to be a much more challenging problem in view of the expression of the curvature form, as convex conjugacy in this generality seems to be a bit less helpful.

Note that these assumptions are satisfied in a large variety of situations. We expect that the second part of assumption (T), in terms of the moment polytope, is actually implied by the assumption that X is toroidal. This is true at least for symmetric varieties and horospherical varieties. Assumption (R) is satisfied for example when the symmetric fiber is of group type, of type AIII(r, $n > 2r$), of type AII(p), but unfortunately not when the symmetric fiber is of type AI(m). It is obviously satisfied if the variety X is horospherical.

Recall that we fixed an anticanonical $\mathbb{Q}$-divisor D^{ac} with a decomposition $D^{ac} = D^{ac}_f + D^{ac}_P$ in Section 5.4 such that $\mathcal{O}(D^{ac}_P) = f^*K^{-1}_{G/P}$ on the complement $\hat{X}$ of codimension ≥ 2 orbits, and $D^{ac}_f = \sum_Y n_Y Y$ where

$$n_Y = 1 - \sum_{\alpha \in \Phi_{Q^U} \cup \Phi_s^+} \alpha \circ \mathcal{P}(\mu_Y).$$

Let Θ be a G-stable $\mathbb{Q}$-divisor on X with simple normal crossing support. Write $\Theta = \sum c_Y Y$ and set

$$n_{Y,\Theta} = -c_Y + 1 - \sum_{\alpha \in \Phi_{Q^U} \cup \Phi_s^+} \alpha \circ \mathcal{P}(\mu_Y).$$

We assume all coefficients c_Y satisfy $c_Y < 1$. Recall that we denote $\sum_{\alpha \in \Phi_{Q^U}} \alpha \circ \mathcal{H}$ by χ^{ac} and this is the isotropy character associated to the anticanonical line bundle on G/H.

Fix a smooth positive reference metric h_{ref} on $\mathcal{L}$, and denote its curvature form by ω_{ref}. Let $\mathrm{rPSH}^K(X, \omega_{\mathrm{ref}})$ denote the space of smooth K-invariant strictly ω_{ref}-plurisubharmonic potentials on X. The functions in $\mathrm{rPSH}^K(X, \omega_{\mathrm{ref}})$ are in one-to-one correspondence with smooth positive hermitian metrics on $\mathcal{L}$. We denote by h_ϕ the metric corresponding to $\phi \in \mathrm{rPSH}^K(X, \omega_{\mathrm{ref}})$ and we write $\omega_\phi = \omega_{\mathrm{ref}} + i\partial\bar{\partial}\phi$ for the curvature of h_ϕ, which depends on ϕ only up to an additive constant.

To any $\phi \in \mathrm{rPSH}^K(X, \omega_{\mathrm{ref}})$ is associated a toric potential u: the toric potential of h_ϕ. Note that under our assumptions (X is smooth and toroidal) and by Proposition 4.17, X admits a smooth toric submanifold Z, and u is the toric potential of the restriction of h_ϕ to the restricted ample $\mathbb{Q}$-line bundle $\mathcal{L}|_Z$, hence the convex potential u^* of u satisfy the Guillemin-Abreu regularity conditions in terms of the polytope $-\Delta^t$.

7.2. **Scalar curvature.** The scalar curvature S of a smooth Kähler form ω is defined by the formula

$$S = \frac{n\mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}$$

Note that $\mathrm{Ric}(\omega)$ is the curvature form of the metric on K_X^{-1} corresponding to the volume form ω^n, whose toric potential we denote by $\tilde{u}$. Assuming that ω is the curvature form of a metric on $\mathcal{L}$, we can determine this toric potential using

Theorem 3.10. Using the liberty to choose the multiplicative constant for the section defining the toric potential (a multiple of the dual of $\bigwedge_\diamond \gamma_\diamond$), we may assume that,

$$\tilde{u}(a) = -\ln\det d_a^2 u - \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} \ln((2\chi - d_a u)(\alpha^\vee)) + \sum_{\beta\in\Phi_s^+} \ln\sinh(-2\beta(a)) - \sum_{\alpha\in\Phi_{Q^u}} 2\alpha(a),$$

for $a \in -\mathfrak{a}_s^+$.

We will, for the rest of the section, fix a choice of orthonormal basis $(l_j)_j$ of $\mathfrak{a}_s$ (with respect to some fixed scalar product whose corresponding Lebesgue measure is normalized by $\mathcal{M}$) and corresponding dual basis (l_j^*) of $\mathfrak{a}_s^*$. We write $\alpha^{\vee,j}$ for the coordinates of $\alpha^\vee$. We use the notations $d_a u = \sum_j u_j(a) l_j^*$, $(u^{j,k})$ for the inverse matrix of $(u_{j,k})$, etc. To simplify notations, we sometimes omit summation symbols in which case we sum over repeated indices in a given term. We then have

$$\tilde{u} = -\ln\det(u_{l,m}) - \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} \ln((2\chi_l - u_l)\alpha^{\vee,l}) + I_H$$

where we set $I_H(a) = \sum_{\beta\in\Phi_s^+} \ln\sinh(-2\beta(a)) - \sum_{\alpha\in\Phi_{Q^u}} 2\alpha(a)$. We set $p = d_a u$ and consider both a and p as variables in the dual spaces $\mathfrak{a}_s$ and $\mathfrak{a}_s^*$.

Proposition 7.1. *The scalar curvature at* $\exp(a)H$ *is equal to*

$$-u_{i,j}^{*,i,j}(p) + \Big(-2u_j^{*,i,j}(p) + (I_H)_i(a)\Big)\frac{P'_{DH,i}}{P'_{DH}}(p) + u_{i,j}^*(p) I_{H,i,j}(a) - u^{*,i,j}(p)\frac{P'_{DH,i,j}}{P'_{DH}}(p) + \sum_{\alpha\in\Phi_{Q^u}} \frac{2\chi^{ac}(\alpha^\vee)}{(2\chi - p)(\alpha^\vee)}$$

Proof. We compute, using Jacobi's formula,

$$\tilde{u}_j = -u^{l,m}u_{m,l,j} - \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} \frac{-u_{l,j}\alpha^{\vee,l}}{(2\chi - u_l)\alpha^{\vee,l}} + I_{H,j}$$

Using the variable $p = d_a u$ and convex conjugate, we have $d_p u^* = a$, $d_a^2 u = (d_p^2 u^*)^{-1}$ and thus $u_j^{*,i,j}(p) = u^{k,j}(a)u_{j,k,i}(a)$. We may then give another expression of $\tilde{u}_j$:

$$\tilde{u}_j(a) = -u_i^{*,j,i}(p) - \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} \frac{-u^{*,l,j}(p)\alpha^{\vee,l}}{(2\chi - p)(\alpha^\vee)} + I_{H,j}(d_p u^*).$$

then

$$\tilde{u}_{j,k}(a) = -u_{i,s}^{*,j,i}(p)u^{*,s,k}(p) - \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} \left(\frac{-u^{*,l,j}\alpha^{\vee,l}}{(2\chi - p)(\alpha^\vee)}\right)_s (p)u^{*,s,k}(p) + I_{H,j,k}(a).$$

We now compute for $a \in -\mathfrak{a}_s^+$,

$$\frac{n\mathrm{Ric}(\omega_\phi)\wedge\omega_\phi^{n-1}}{\omega_\phi^n} = \mathrm{Tr}((\tilde{u}_{l,m}(a))(u^{l,m}(a))) + \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\frac{-\tilde{u}_l(a)\alpha^{\vee,l}}{(2\chi-p)(\alpha^\vee)} + \sum_{\alpha\in\Phi_{Q^u}}\frac{2\chi^{ac}(\alpha^\vee)}{(2\chi-p)(\alpha^\vee)}$$

which, by using the previous expressions, is equal to

$$-u^{*,i,j}_{i,j} + \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} 2\frac{u^{*,i,j}_j\alpha^{\vee,i}}{(2\chi-p)(\alpha^\vee)} - \sum_{\alpha,\beta\in\Phi_{Q^u}\cup\Phi_s^+}\frac{u^{*,i,j}\alpha^{\vee,i}\beta^{\vee,j}}{(2\chi-p)(\alpha^\vee)(2\chi-p)(\beta^\vee)}$$
$$- \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\frac{u^{*,i,j}\alpha^{\vee,i}\alpha^{\vee,j}}{((2\chi-p)(\alpha^\vee))^2} + u^*_{i,j}(I_H)_{i,j}(a)$$
$$- \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\frac{(I_H)_i(a)\alpha^{\vee,i}}{(2\chi-p)(\alpha^\vee)} + \sum_{\alpha\in\Phi_{Q^u}}\frac{2\chi^{ac}(\alpha^\vee)}{(2\chi-p)(\alpha^\vee)}$$

Let $P'_{DH}(p) = P_{DH}(2\chi-p)$, then note that

$$P'_{DH,i}(p) = P'_{DH}(p)\sum_\alpha \frac{-\alpha^{\vee,i}}{(2\chi-p)(\alpha^\vee)}$$

and

$$P'_{DH,i,j}(p) = P'_{DH}(p)\left(\sum_{\alpha,\beta}\frac{\alpha^{\vee,i}\beta^{\vee,j}}{(2\chi-p)(\alpha^\vee)(2\chi-p)(\beta^\vee)} + \sum_\alpha\frac{\alpha^{\vee,i}\alpha^{\vee,j}}{(2\chi-p)(\alpha^\vee)^2}\right).$$

Plugging this into the last expression of the scalar curvature yields the result. □
□

Remark 7.2. The computation of the scalar curvature here is only on the homogeneous space, hence holds under weaker hypothesis than in the setting: we only need to assume that $\mathcal{L}$ is a line bundle on G/H whose restriction to the symmetric fiber is trivial, and that h is a smooth and positive metric on $\mathcal{L}$.

Denote by $\bar{S}$ the average scalar curvature, defined as

$$\bar{S} = \frac{\int_X n\mathrm{Ric}(\omega)\wedge\omega^{n-1}}{\int_X\omega^n}.$$

7.3. The functionals.

7.3.1. *The J-functional.* The J-functional is defined (up to a constant) on the space $\mathrm{rPSH}^K(X,\omega_{\mathrm{ref}})$ by its variations as follows: if ϕ_t is a smooth path in $\mathrm{rPSH}^K(X,\omega_{\mathrm{ref}})$ between the origin and ϕ, then

$$J(\phi) = \int_0^1\int_X\dot{\phi}_t\frac{\omega_{\mathrm{ref}}^n-\omega_{\phi_t}^n}{(2\pi)^n\mathcal{L}^n}dt.$$

Proposition 7.3. *Let u denote the toric potential of h_ϕ. Then*

$$\left|J(\phi) - u(0) - \frac{n!}{\mathcal{L}^n}\int_{\Delta^+}u^*(2\chi-2q)P_{DH}(q)dq\right|$$

is bounded independently of ϕ.

Proof. We have

$$J(\phi)=\int_X\phi\frac{\omega_{\mathrm{ref}}^n}{(2\pi)^n\mathcal{L}^n}-\int_0^1\int_X\dot\phi_t\frac{\omega_{\phi_t}^n}{(2\pi)^n\mathcal{L}^n}dt$$

The definition of convex conjugate yields $\dot u_t^*(d_au)=-\dot u_t(a)$ hence by Proposition 6.9 and Corollary 6.10, we have

$$\begin{aligned}\int_0^1\int_X\dot\phi_t\omega_{\phi_t}^n dt&=-\int_0^1 C'_H\int_{\chi+\Delta^t\cap\bar C^+}\dot u_t^*(2\chi-2q)P_{DH}(q)dq\\&=-C'_H\int_{\chi+\Delta^t\cap\bar C^+}(u^*-u^*_{\mathrm{ref}})(2\chi-2q)P_{DH}(q)dq\\&=-(2\pi)^n n!\int_{\Delta^+}(u^*-u^*_{\mathrm{ref}})(2\chi-2q)P_{DH}(q)dq\end{aligned}$$

On the other hand, it follows from classical results [GZ05, Proposition 2.7] that

$$\left|\frac{1}{(2\pi)^n\mathcal{L}^n}\int_X\phi\omega_{\mathrm{ref}}^n-\sup_X(\phi)\right|$$

is bounded independently of $\phi\in\mathrm{rPSH}^K(X,\omega_{ref})$. We have

- $\sup_X(\phi)=\sup_{\mathfrak{a}_s}(u-u_{\mathrm{ref}})$,
- $w_{-2\Delta^t}-C_1\leq u_{\mathrm{ref}}\leq w_{-2\Delta^t}+u_{\mathrm{ref}}(0)$ for some constant C_1 by Proposition 6.3, and
- $\sup_{\mathfrak{a}_s}u-w_{-2\Delta^t}=u(0)$ by convexity,

hence

$$\left|\int_X\phi\omega_{\mathrm{ref}}^n-(2\pi)^n\mathcal{L}^n u(0)\right|$$

is bounded independently of ϕ. □ □

7.3.2. *Mabuchi functional.* The (log)-Mabuchi functional Mab_Θ relative to Θ is defined also by integration along smooth path: $\mathrm{Mab}_\Theta(\phi)$ is equal to

$$\int_0^1\Big\{\int_X\dot\phi_t(\bar S_\Theta\omega_{\phi_t}-n\mathrm{Ric}(\omega_{\phi_t}))\wedge\frac{\omega_{\phi_t}^{n-1}}{(2\pi\mathcal{L})^n}+2\pi n\sum_Y c_Y\int_Y\dot\phi_t\frac{\omega_{\phi_t}^{n-1}}{(2\pi\mathcal{L})^n}\Big\}dt$$

where $\bar S_\Theta=\bar S-n\sum_Y c_Y\mathcal{L}|_Y^{n-1}/\mathcal{L}^n$.

Let $\tilde\Delta_Y^+$ denote the bounded cone with vertex λ_0 and base Δ_Y^+. Note that in general $\Delta^+\neq\bigcup_Y\tilde\Delta_Y^+$. In is however the case under assumption (T), that is if we assume that X is toroidal.

We set $\Lambda_Y=(n_Y-c_Y)/v_{\mathcal{L}}(\mu_Y)$.

Theorem 7.4. *Under assumption (T), up to the choice of a normalizing additive constant,* $\frac{\mathcal{L}^n}{n!}\mathrm{Mab}_\Theta(\phi)$ *is equal to*

$$\begin{aligned}&\sum_Y\Lambda_Y\int_{\tilde\Delta_Y^+}\Big(nu^*(p)-u^*(p)\sum\frac{\chi(\alpha^\vee)}{q(\alpha^\vee)}+d_pu^*(p)\Big)P_{DH}(q)dq\\&+\int_{\Delta^+}u^*(p)\Big(\sum\frac{\chi^{ac}(\alpha^\vee)}{q(\alpha^\vee)}-\bar S_\Theta\Big)P_{DH}(q)dq-\int_{\Delta^+}I_H(a)P_{DH}(q)dq\\&-\int_{\Delta^+}\ln\det(u^*_{i,j})(p)P_{DH}(q)dq\end{aligned}$$

Proof. The summands

$$\int_0^1\int_X \dot\phi_t \bar S_\Theta \omega_{\phi_t}^n dt/(2\pi\mathcal{L})^n$$

and

$$2\pi n\sum_Y c_Y\int_Y \dot\phi_t\omega_{\phi_t}^{n-1}dt/(2\pi\mathcal{L})^n$$

may be dealt with as in the computation for J, yielding respectively

$$\bar S_\Theta\frac{n!}{\mathcal{L}^n}\int_{\Delta^+}(u^*_{\mathrm{ref}}-u^*)(2\chi-2q)P_{DH}(q)dq$$

and

$$\sum_Y c_Y\frac{n!}{\mathcal{L}^n}\int_{\Delta^+_Y}(u^*_{\mathrm{ref}}-u^*)(2\chi-2q)P_{DH}(q)dq_Y.$$

The harder part is

$$\int_0^1\int_X -n\dot\phi_t\mathrm{Ric}(\omega_{\phi_t})\wedge\omega_{\phi_t}^{n-1}/(2\pi\mathcal{L})^n dt.$$

Using the integration formula as well as the formula for the scalar curvature, we have

$$\begin{aligned} I :=&\int_X -n\dot\phi_t\mathrm{Ric}(\omega_{\phi_t})\wedge\omega_{\phi_t}^{n-1} = \int_X -n\dot\phi_t\frac{\mathrm{Ric}(\omega_{\phi_t})\wedge\omega_{\phi_t}^{n-1}}{\omega_{\phi_t}^n}\omega_{\phi_t}^n \\ =&\frac{C'_H}{2^n}\int_{\Delta'}\dot u_t^*\Big(-u^{*,i,j}_{t,i,j}P'_{DH}-2u^{*,i,j}_{t,j}P'_{DH,i}-u_t^{*,i,j}P'_{DH,i,j} \\ &+u^*_{t,i,j}I_{H,i,j}(a)P'_{DH}+I_{H,i}(a)P'_{DH,i}+\sum_{\alpha\in\Phi_{Q^u}}\frac{2\chi^{ac}(\alpha^\vee)}{(2\chi-p)(\alpha^\vee)}P'_{DH}\Big)dp \end{aligned}$$

where the variable, if omitted, is $p=d_au$.

Apart from the last term, the situation is analogous to Li-Zhou-Zhu [LZZ] hence we may follow the same steps. However note that translation by χ in the Duistermaat-Heckman polynomial will sometimes introduce extra terms. Denote by ν the unit outer normal to $\partial\Delta'$. On a codimension one face corresponding to a facet Y of X, ν is a positive multiple of μ_Y.

We will apply several times the divergence theorem as follows, without writing the details every time. For $0<s<1$, let Δ'_s denote the bounded cone with vertex $2(\chi-\lambda_0)$ and base the dilation by s of the boundary $s\partial\Delta'$. We may apply the divergence theorem on Δ'_s to smooth functions on the interior of $-2\Delta^t$, then take the limit as $s\to 1$, applying dominated convergence and using the appropriate convergence result. We let $d\sigma$ denote the area measure on all boundaries.

It follows from Section 6.3 that for $p\in\Delta'_Y$ and any i,

$$\lim_{s\to 1}u_{i,j}(d_{sp}u^*)\nu_j=0$$

and

$$\lim_{s\to 1}\tilde u_j(d_{sp}u^*)(\mu_Y)_j=n_Y.$$

Recall that

$$\tilde u_j(a)=-u_i^{*,j,i}(p)-\sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+}\frac{-u^{*,l,j}(p)\alpha^{\vee,l}}{(2\chi-p)(\alpha^\vee)}+I_{H,j}(d_pu^*).$$

We deduce from these facts, and the fact that P'_{DH} vanishes on restricted Weyl walls, that

$$\lim_{s\to 1}\int_{\partial\Delta'_s}\dot{u}^*_t(-u^{*,i,j}_{t,j}+I_{H,i}(a))\nu_i P'_{DH}d\sigma=\sum_Y\int_{\Delta'_Y}2n_Y\frac{\nu}{\mu_Y}\dot{u}^*_t P'_{DH}d\sigma$$

where Δ'_Y denotes the facet of Δ' whose outer normal is μ_Y.

A first application of the divergence theorem then yields, by passing to the limit, that the above quantity is equal to

$$\int_{\Delta'}\Big(\dot{u}^*_t(-u^{*,i,j}_{t,j}+I_{H,i}(a))P'_{DH}\Big)_i dp.$$

We may compute

$$\begin{aligned}\Big(\dot{u}^*_t(-u^{*,i,j}_{t,j}+I_{H,i}(a))P'_{DH}\Big)_i =& -\dot{u}^*_{t,i}u^{*,i,j}_{t,j}P'_{DH}+\dot{u}^*_{t,i}I_{H,i}(a)P'_{DH}\\ &-\dot{u}^*_t u^{*,i,j}_{t,i,j}P'_{DH}+\dot{u}^*_t u^*_{j,i}I_{H,i,j}(a)P'_{DH}\\ &-\dot{u}^*_t u^{*,i,j}_{t,j}P'_{DH,i}+\dot{u}^*_t I_{H,i}(a)P'_{DH,i}\end{aligned}$$

and

$$\dot{u}^*_{t,i}I_{H,i}(a)P'_{DH}=\frac{d}{dt}(I_H(a)P'_{DH}).$$

Consider now the vector field $(\dot{u}^*_{t,i}P'_{DH}-\dot{u}^*_t P'_{DH,i})u^{*,i,j}_t$. Applying the divergence theorem to this vector field yields

$$0=\int_{\Delta'}(\dot{u}^*_{t,i,j}u^{*,i,j}_t P'_{DH}+\dot{u}^*_{t,i}u^{*,i,j}_{t,j}P'_{DH}-\dot{u}^*_t u^{*,i,j}_{t,j}P'_{DH,i}-\dot{u}^*_t u^{*,i,j}_t P'_{DH,i,j})dp$$

Note here that

$$\dot{u}^*_{t,i,j}u^{*,i,j}_t=\frac{d}{dt}(\ln\det(u^*_{t,i,j})).$$

From these two applications of the divergence theorem and the expression of the scalar curvature, we deduce by taking the sum that I is $C'_H/2^n$ times *the derivative with respect to* t of

$$\begin{aligned}&\sum_Y\int_{\Delta'_Y}2n_Y\frac{\nu}{\mu_Y}u^*_t P'_{DH}d\sigma-\int_{\Delta'}I_H(a)P'_{DH}dp-\int_{\Delta'}\ln\det(u^*_{t,i,j})P'_{DH}dp\\ &+\int_{\Delta'}u^*_t\sum_{\alpha\in\Phi_{Q^u}}\frac{2\chi^{ac}(\alpha^\vee)}{(2\chi-p)(\alpha^\vee)}P'_{DH}dp\end{aligned}$$

hence the value of the above expression at $t=1$ is the corresponding term of the Mabuchi functional, up to a constant independent of ϕ.

We finally use the divergence theorem one last time, applied to the vector field $u^*_t p_i P'_{DH}$ to obtain, for every Y,

$$\begin{aligned}\int_{\Delta'_Y}p_i\nu_i u^*_t P'_{DH}d\sigma &=\int_{\tilde{\Delta}'_Y}(u^*_t P'_{DH,i}p_i+ru^*_t P_{DH}+u^*_{t,i}p_i P'_{DH})dp\\ &=\int_{\tilde{\Delta}'_Y}(nu^*_t-u^*_t\sum_\alpha\frac{2\chi_i\alpha^{\vee,i}}{(2\chi-p)(\alpha^\vee)}+u^*_{t,i}p_i)P'_{DH}dp\end{aligned}$$

Since $2n_Y\frac{\nu}{\mu_Y}=\frac{n_Y}{v_{\mathcal{L}}(\mu_Y)}p(\nu)$ and $dp_Y=\frac{p(\nu)}{v_{\mathcal{L}}(\mu_Y)}d\sigma$, where dp_Y is the image of the measure dq_Y under translation by $-\chi$ then dilation by 2. This allows to transform the remaining integrals on Δ'_Y or Δ^+_Y to integrals on Δ^+, after the change of variable from p to q. Putting everything together gives the result. □ □

Remark 7.5. As a corollary of the proof, by applying the same steps, we can compute $\bar{S}_\Theta$. We obtain

$$\bar{S}_\Theta = \sum_Y \int_{\tilde{\Delta}_Y^+} (n\Lambda_Y P_{DH}(q) + d_q P_{DH}(\chi^{ac} - \Lambda_Y \chi))dq \Big/ \int_{\Delta^+} P_{DH}(q)dq$$

7.4. **Action of** $Z(L)^0$**.** Following Donaldson [Don02], let us write the Mabuchi functional as the sum of a linear and non-linear part $\mathrm{Mab}_\Theta = \mathrm{Mab}_\Theta^l + \mathrm{Mab}_\Theta^{nl}$ by setting

$$\begin{aligned}\frac{\mathcal{L}^n}{n!}\mathrm{Mab}_\Theta^l(\phi) = &\sum_Y \Lambda_Y \int_{\tilde{\Delta}_Y^+} (nu^*(p) - u^*(p)\sum \frac{\chi(\alpha^\vee)}{q(\alpha^\vee)} + d_p u^*(p))P_{DH}dq \\ &+ \int_{\Delta^+} u^*(p)(\sum \frac{\chi^{ac}(\alpha^\vee)}{q(\alpha^\vee)} - \bar{S}_\Theta)P_{DH}dq + \int_{\Delta^+} 4\rho_H(a)P_{DH}(q)dq\end{aligned}$$

where $2\rho_H = \sum_{\alpha\in\Phi_{Q^u}\cup\Phi_s^+} \alpha\circ\mathcal{P}$. We will use the notation $M^l(u^*)$ to denote the above as a function of u^*, where u^* is the convex conjugate of the toric potential of h_ϕ. Similarly, we use the notation $M^{nl}(u^*) = \frac{\mathcal{L}^n}{n!}\mathrm{Mab}_\Theta(\phi) - M^l(u^*)$.

Remark 7.6. In the last step of the proof of Theorem 7.4, if we apply the divergence theorem to the vector field $4u^*P'_{DH}\rho_H$ instead of $u_t^* p_i P'_{DH}$ we obtain another expression of the linear part of the Mabuchi functional $\frac{\mathcal{L}^n}{n!}\mathrm{Mab}_\Theta^l(\phi)$:

$$\begin{aligned}M^l(u^*) = &\sum_Y \frac{1-c_Y}{v_{\mathcal{L}}(\mu_Y)} \int_{\Delta'_Y} p(\nu)u^*(p)P'_{DH}(p)d\sigma - \int_{\Delta^+} d_pP'_{DH}(4\rho_H)u^*(p)dp \\ &+ \int_{\Delta^+} u^*(p)(\sum \frac{\chi^{ac}(\alpha^\vee)}{q(\alpha^\vee)} - \bar{S}_\Theta)P_{DH}dq.\end{aligned}$$

7.4.1. *Invariance of Mabuchi functional and log-Futaki invariant.* Consider the connected center $\mathcal{Z}(L)^0$ of L. It acts on the right on G/H and the action extends to X. The induced action on K-invariant singular hermitian metrics on $\mathcal{L}$ stabilizes the set $\mathrm{rPSH}^K(X, \omega_{\mathrm{ref}})$. More precisely, for $b \in \mathfrak{a}_s \cap \mathfrak{z}(\mathfrak{l})$, and h a K-invariant, non-negatively curved singular hermitian metric on $\mathcal{L}$ with toric potential u, the toric potential of the image by $\exp(b)$ of h is $a \mapsto u(a+b)$. This translates on convex conjugates as replacing u^* by $u_b^* = u^* - b$. Note that it is enough to consider only elements of $\mathfrak{a}_s \cap \mathfrak{z}(\mathfrak{l})$ since $\mathcal{Z}(L)^0 = T \cap \mathcal{Z}(L)^0$, and $T\cap H$ as well as $T \cap K$ act trivially. Since $du_b^* = du^* - b$, $\alpha(b) = 0$ for $\alpha \in \Phi_L$, $\chi(b) = 0$ and $d^2u_b^* = d^2u^*$, we have the following proposition.

Proposition 7.7. *The Mabuchi functional is invariant under the action of $\mathcal{Z}(L)^0$ on the right if and only if*

$$\begin{aligned}0 = &\sum_Y \int_{\tilde{\Delta}_Y^+} -2q(b)\big(((n+1)\Lambda_Y - \bar{S}_\Theta)P_{DH}(q) + d_qP_{DH}(\chi^{ac} - \chi)\big)dq \\ &+ \int_{\Delta^+} 2\sum_{\alpha\in\Phi_{Q^u}} \alpha(b)P_{DH}(q)dq\end{aligned}$$

for all $b \in \mathfrak{a}_s \cap \mathfrak{z}(\mathfrak{l})$.

The expression in the theorem could naturally be interpreted as a log Calabi-Futaki invariant.

7.4.2. *Normalization of potentials.* The action of $Z(L)^0$ allows on the other hand to normalize the psh potentials, as follows. Given $\phi \in \mathrm{rPSH}(X, \omega_{ref})$, we may add a constant and use the action of an element of $Z(L)^0$ to obtain another potential $\hat{\phi} \in \mathrm{rPSH}(X, \omega_{ref})$, such that if $\hat{u}$ is the corresponding toric potential, and $\hat{u}^*$ its convex conjugate, we have $\min_{-2\Delta^t} \hat{u}^* = \hat{u}^*(2(\chi - \lambda_0)) = 0$.

7.5. **Coercivity criterion.**

7.5.1. *Statement of the criterion and examples.*

Definition 7.8. The Mabuchi functional is *proper modulo the action of* $Z(L)^0$ if there exists positive constants ϵ and C such that for any $\phi \in \mathrm{rPSH}(X, \omega_{ref})$, there exists $g \in Z(L)^0$ such that

$$\mathrm{Mab}_\Theta(\phi) \geq \epsilon J(g \cdot \phi) - C.$$

Consider the function $F_{\mathcal{L}}$ defined piecewise by

$$F_{\mathcal{L}}(q) = (n+1)\Lambda_Y - \bar{S}_\Theta + \sum \frac{(\chi^{ac} - \Lambda_Y \chi)(\alpha^\vee)}{q(\alpha^\vee)}$$

for q in the unbounded cone with vertex $\lambda_0 - \chi$ and generated by $\tilde{\Delta}_Y^+$. Note that $F_{\lambda\mathcal{L}}(q) = \frac{1}{\lambda} F_{\mathcal{L}}(q)$. The Mabuchi functional for the line bundle $\mathcal{L}$ is proper if and only if it is proper for any positive rational multiple of $\mathcal{L}$. As an application of this remark, if $F_{\mathcal{L}} > 0$, we may choose the multiple $\lambda\mathcal{L}$ in such a way that

$$\int_{\Delta^+} P_{DH} dq = \sum_Y \int_{\tilde{\Delta}_Y^+} F_{\lambda\mathcal{L}}(q) P_{DH}(q) dq.$$

We now replace $\mathcal{L}$ by its multiple to assume the equality above is satisfied with $\lambda = 1$. We then define a barycenter of Δ^+ by:

$$\mathrm{bar} = \sum_Y \int_{\tilde{\Delta}_Y^+} q F_{\mathcal{L}}(q) P_{DH}(q) dq / \int_{\Delta^+} P_{DH} dq.$$

Theorem 7.9. *Assume the following:*

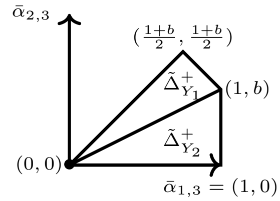
- $F_{\mathcal{L}} > 0$,
- $(\min_Y \Lambda_Y)(\mathrm{bar} - \chi) - 2\rho_H$ *is in the relative interior of the dual cone of* $\mathfrak{a}_s^+$,
- *assumptions (T) and (R) are satisfied.*

Then the Mabuchi functional is proper modulo the action of $Z(L)^0$.

Example 7.10. We have determined in previous examples everything that is necessary to check when the criterion apply for the example of the wonderful compactification X of, say, the symmetric space of type AIII(2, 5). We consider the ample Cartier divisors $(1+b)Y_1 + Y_2$ for $0 < b < 1$ rational, and corresponding uniquely G-linearized $\mathbb{Q}$-line bundles. They run over all ample divisors on X that are trivial on the open orbit, up to rational multiple. We illustrate in Figure 3 the corresponding polytopes and subdivision by $\tilde{\Delta}_{Y_1}^+$ and $\tilde{\Delta}_{Y_2}^+$. Then it is easy, with computer assistance, to check when the criterion is satisfied in terms of b, and we obtain bounds $b^- \simeq 0.31$ and $b^+ \simeq 0.54$ such that when $b^- < b < b^+$, the Mabuchi functional (for $\mathcal{L} = \mathcal{O}((1+b)Y_1 + Y_2)$ and $\Theta = 0$) is proper modulo the action of $Z(L)^0$.

Remark 7.11. The two first assumptions imply readily, from the last section, that the Mabuchi functional is invariant under the action of $Z(L)^0$.

FIGURE 3. Moment polytopes for type AIII(2, 5)



Remark 7.12. In the case of group compactifications, assumption (R) is automatically satisfied, and we may use [LZZ] in the later stages of the proof to remove assumption (T).

If $\mathcal{L} = K^{-1}_{X,\Theta}$ then $\Lambda_Y = 1$ for all Y, $\bar{S}_\Theta = n$, and $\chi = \chi^{ac}$ (up to changing the linearization of $\mathcal{L}$ by a character of G). Furthermore,

$$\chi^{ac} + 2\rho_H = \sum_{\alpha \in \Phi_{Q^u} \cup \Phi_s^+} \alpha \circ \mathcal{H} + \sum_{\alpha \in \Phi_{Q^u} \cup \Phi_s^+} \alpha \circ \mathcal{P} = \sum_{\alpha \in \Phi_{Q^u} \cup \Phi_s^+} \alpha.$$

We then have the corollary:

Corollary 7.13. *If $\mathcal{L} = K^{-1}_{X,\Theta}$ then the Mabuchi functional is proper modulo the action of $Z(L)^0$ if and only if assumptions (T) and (R) are satisfied and the translated barycenter* bar $- \sum_{\alpha \in \Phi_{Q^u} \cup \Phi_s^+} \alpha$ *is in the relative interior of the dual cone of $\mathfrak{a}_s^+$.*

Remark 7.14. This generalizes the criterion for existence of Kähler-Einstein metrics obtained in [Del] in the sufficient direction. In fact the condition is also necessary in this case, as would follow from a computation of log-Donaldson-Futaki invariants along special equivariant test configurations for example.

Remark 7.15. Again, for group compactifications, even assumption (T) may be removed by using [LZZ] in the later stages of the proof.

We prove Theorem 7.9 in the following subsections so unless otherwise stated we make the assumptions as in the statement.

7.5.2. *Inequality for* Mab^l_Θ. Assume $u^* = \hat{u}^*$ is normalized as in Section 7.4.2. Note the following elementary lemma, following directly from the convexity and normalization of $\hat{u}^*$.

Lemma 7.16. *Assume u^* is normalized, then*

$$\int_{\Delta'} u^*(p) P'_{DH}(p) dp \leq C_\partial \int_{\partial\Delta'} u^* P'_{DH}(p) d\sigma$$

for some constant C_∂ independent of u^*.

Proposition 7.17. *Under the assumptions of Theorem 7.9, there exists a positive constant C_l such that for any normalized u^*,*

$$M^l(u^*) \geq C_l \int_{\partial\Delta'} u^* P'_{DH}(p) d\sigma.$$

Proof. Suppose there exists a sequence of normalized u_j^* such that

$$\int_{\partial\Delta'} u_j^* P'_{DH}(p) d\sigma = 1$$

and $M^l(u)$ decreases to 0. By a compactness theorem proved by Donaldson [Don08, Section 5.2], we may assume that u_j^* converges locally uniformly on $-2\Delta^t$ to a convex function v_∞, still satisfying $v_\infty(2(\chi-\lambda_0)) = \min v_\infty = 0$.

Let $p_0 = 2\chi - 2\text{bar}$. By convexity of u^* we have $d_p u^*(p-p_0) \geq u^*(p) - u^*(p_0)$, hence

$$\begin{aligned}
\mathrm{M}^l(u^*) \geq & \sum_Y \int_{\tilde{\Delta}'_Y} F_{\mathcal{L}}(q)(u^*(p) - u^*(p_0) - d_{p_0}u^*(p-p_0))P'_{DH}(p)dp \\
& + \sum_Y \int_{\tilde{\Delta}'_Y} d_p u^*(\Lambda_Y p_0 + 4\rho_H)P'_{DH}(p)dp \\
& + \sum_Y \int_{\tilde{\Delta}'_Y} F_{\mathcal{L}}(q) d_{p_0} u^*(p-p_0)P'_{DH}(p)dp \\
& + \sum_Y \int_{\tilde{\Delta}'_Y} \left(n\Lambda_Y - \bar{S}_\Theta + \sum \frac{(\chi^{ac} - \Lambda_Y \chi)(\alpha^\vee)}{q(\alpha^\vee)} \right) u^*(p_0) P'_{DH}(p)dp
\end{aligned}$$

The last summand vanishes by the expression of $\bar{S}_\Theta$ obtained in Remark 7.5. The third summand, on the other hand, vanishes by definition of p_0. The second term is non-negative by the assumptions of Theorem 7.9 and the first term is non-negative by convexity of u^*.

Then the fact that $M^l(u_j)$ converges to zero implies that v_∞ is an affine function by the first term, and that its linear part is given by an element of $\mathfrak{Y}(T_s/[L,L])$ by the second term. Since v_∞ is normalized and $2(\chi - \lambda_0)$ is in the interior of $\mathfrak{Y}(T_s/Z(L)) \cap \Delta'$, this means that $v_\infty = 0$. As a consequence, we have

$$\int_{\Delta'} u_j^*(p) P'_{DH}(p) dp \to 0.$$

Let $\delta = \min_Y (1-c_Y)/v_{\mathcal{L}}(\mu_Y)$, it is positive by assumption. By the expression of M^l given in Remark 7.6, and using again that u_j^* converges to 0, we obtain that for j large enough,

$$M^l(u_j^*) \geq \delta \int_{\partial\Delta'} u_j^* P'_{DH}(p) d\sigma = \delta > 0,$$

which provides a contradiction hence proves the proposition. □ □

Remark 7.18. Assumption (R) was not used at all here.

7.5.3. *Inequality for* Mab_Θ^{nl}. The strategy to transfer the coercivity result on the linear part to the full fonctional now follows a general strategy already used by Donaldson in [Don02]. The first step is to get a rather weak estimate on the non-linear part.

Proposition 7.19. *There exists uniform positive constants C_1, C_2, C_3, such that*

$$M^{nl}(u^*) \geq -C_1 \int_{\Delta'} u^*(p) P'_{DH}(p) dp - C_2 \int_{\partial\Delta'} u^*(p) dp - C_3$$

Proof. First note that $-I_H(a) - 4\rho_H(a) \geq 0$ for $a \in -\mathfrak{a}_s^+$. Using this inequality and the convexity of $-\ln\det$, we have

$$M^{nl}(u^*) \geq -\int_{\Delta'} u_{\mathrm{ref}}^{*,i,j} u_{i,j}^* P'_{DH} dp - C_2$$

We now apply the divergence theorem to the vector field

$$u_{\mathrm{ref},i}^{*,i,j} u^* P'_{DH} - u_{\mathrm{ref}}^{*,i,j} u_i^* P'_{DH} + u_{\mathrm{ref}}^{*,i,j} u^* P'_{DH,j}$$

to obtain

$$\sum_Y \int_{\Delta'_Y} u^* P'_{DH} d\sigma = \int_{\Delta'} \Big(u^* u_{\mathrm{ref},i,j}^{*,i,j} P'_{DH} + u^* 2 u_{\mathrm{ref},j}^{*,i,j} P'_{DH,i} + u^* u_{\mathrm{ref}}^{*,i,j} P'_{DH,i,j} - u_{\mathrm{ref}}^{*,i,j} u_{i,j}^* P'_{DH} \Big) .dp$$

Here we used assumption (R) to check that $P'_{DH,j}$ vanishes on restricted Weyl walls. Note that $u_{\mathrm{ref},i,j}^{*,i,j}$ is bounded. Finally, since $u_{\mathrm{ref},j}^{*,i,j} P'_{DH,i}$ and $u_{\mathrm{ref}}^{*,i,j} P'_{DH,i,j}$ are bounded, and thanks to assumption (T), we may apply [LZZ, Lemma 4.6] to obtain the statement (Even though [LZZ, Lemma 4.6] is proved for the particular case of P'_{DH} coming from a group compactification, it applies much more generally for a positive measure with a product of linear functions as density and polytope contained in a chamber defined by these linear functions). □ □

7.5.4. *Conclusion of the proof.*

Proof of Theorem 7.9. Let $0 < \epsilon < 1$. We have $M^{nl}(u^*) \geq M^{nl}(\epsilon u) - C_4$ for some constant C_4, with the obvious definition for $M^{nl}(\epsilon u)$ (it is not important here that ϵu^* does not appear as the convex conjugate of the toric potential of a smooth metric). Furthermore, the proof of Proposition 7.19 applies just as well to ϵu^* and we obtain

$$\begin{aligned}
M(u^*) &\geq M^l(u^*) - \epsilon(C_1 \int_{\Delta'} u^*(p) P'_{DH}(p) dp + C_2 \int_{\partial\Delta'} u^*(p) dp + C_3) - C_4 \\
&\geq \epsilon \int_{\Delta'} u^* P'_{DH} + M^l(u^*) - \epsilon\Big((C_1+1) \int_{\Delta'} u^* P'_{DH} \\
&\quad + C_2 \int_{\partial\Delta'} u^*(p) dp + C_3\Big) - C_4 \\
&\geq \epsilon \int_{\Delta'} u^* P'_{DH} + M^l(u^*) - \epsilon((C_1+1) C_\partial C_l + C_2 C_l) M^l(u^*) \\
&\quad - \epsilon C_3 - C_4 \\
&\geq \epsilon \int_{\Delta'} u^* P'_{DH} - \epsilon C_3 - C_4,
\end{aligned}$$

by choosing $\epsilon = ((C_1+1)C_\partial C_l + C_2 C_l)^{-1}$.

This is enough to conclude: let $\phi \in \mathrm{rPSH}^K(X, \omega_{\mathrm{ref}})$, let $\hat{\phi} = g \cdot \phi + C$ be the normalization of ϕ, obtained *via* the action of some $g \in Z(L)^0$ and addition of a constant, then since the assumptions imply that the Mabuchi functional is invariant under the action of $Z(L)^0$, we have $\mathrm{Mab}(\phi) = \mathrm{Mab}(\hat{\phi})$. By Proposition 7.3, since the toric potential of a normalized $\hat{\phi}$ satisfies $u(0) = 0$, we have $\int_{\Delta'} \hat{u}^* P'_{DH} \geq$

$\frac{\mathcal{L}^n}{n!}J(\hat{\phi}) - C_5$ for some constant C_5 independent of ϕ. Hence we have

$$\begin{aligned}\mathrm{Mab}_\Theta(\phi) &= \frac{n!}{\mathcal{L}^n} M^l(\hat{u}^*)\\ &\geq \epsilon \frac{n!}{\mathcal{L}^n}\int_{\Delta'} \hat{u}^* P'_{DH} - \frac{n!}{\mathcal{L}^n}(\epsilon C_3 + C_4)\\ &\geq \epsilon J(\hat{\phi}) - \frac{n!}{\mathcal{L}^n}(C_5 + \epsilon C_3 + C_4)\end{aligned}$$

□ □

References

[AB04a] Valery Alexeev and Michel Brion. Stable reductive varieties. I. Affine varieties. *Invent. Math.*, 157(2):227–274, 2004.

[AB04b] Valery Alexeev and Michel Brion. Stable reductive varieties. II. Projective case. *Adv. Math.*, 184(2):380–408, 2004.

[AK05] Valery Alexeev and Ludmil Katzarkov. On K-stability of reductive varieties. *Geom. Funct. Anal.*, 15(2):297–310, 2005.

[Bif90] Emili Bifet. On complete symmetric varieties. *Adv. Math.*, 80(2):225–249, 1990.

[BP87] Michel Brion and Franz Pauer. Valuations des espaces homogènes sphériques. (Valuations of spherical homogeneous spaces). *Comment. Math. Helv.*, 62:265–285, 1987.

[Bri] Michel Brion. Variétés sphériques. Notes de la session de la S. M. F. "Opérations hamiltoniennes et opérations de groupes algébriques".

[Bri89] Michel Brion. Groupe de Picard et nombres caractéristiques des variétés sphériques. *Duke Math. J.*, 58(2):397–424, 1989.

[Bri90] Michel Brion. Vers une généralisation des espaces symétriques. *J. Algebra*, 134(1):115–143, 1990.

[Bri91] Michel Brion. Sur la géométrie des variétés sphériques. *Comment. Math. Helv.*, 66(2):237–262, 1991.

[Bri97] Michel Brion. Curves and divisors in spherical varieties. In *Algebraic groups and Lie groups. A volume of papers in honour of the late R. W. Richardson*, pages 21–34. Cambridge: Cambridge University Press, 1997.

[CC18] X. Chen and J. Cheng. On the constant scalar curvature K\"ahler metrics, general automorphism group. *ArXiv e-prints*, January 2018.

[Dar17] Tamás Darvas. Metric Geometry of Normal Kähler Spaces, Energy Properness, and Existence of Canonical Metrics . *IMRN*, (22):6752–6777, 2017.

[Del] Thibaut Delcroix. K-Stability of Fano sperical varieties. arXiv:1608.01852.

[Del17] Thibaut Delcroix. Kähler-Einstein metrics on group compactifications. *Geom. Funct. Anal.*, 27(1):78–129, 2017.

[Don02] Simon K. Donaldson. Scalar curvature and stability of toric varieties. *J. Differ. Geom.*, 62(2):289–349, 2002.

[Don08] Simon K. Donaldson. Kähler geometry on toric manifolds, and some other manifolds with large symmetry. In *Handbook of geometric analysis. No. 1*, volume 7 of *Adv. Lect. Math. (ALM)*, pages 29–75. Int. Press, Somerville, MA, 2008.

[DP83] Corrado De Concini and Claudio Procesi. Complete symmetric varieties. In *Invariant theory (Montecatini, 1982)*, volume 996 of *Lecture Notes in Math.*, pages 1–44. Springer, Berlin, 1983.

[DP85] Corrado De Concini and Claudio Procesi. Complete symmetric varieties. II. Intersection theory. In *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, volume 6 of *Adv. Stud. Pure Math.*, pages 481–513. North-Holland, Amsterdam, 1985.

[FJ80] Mogens Flensted-Jensen. Discrete series for semisimple symmetric spaces. *Ann. of Math. (2)*, 111(2):253–311, 1980.

[GH15] Giuliano Gagliardi and Johannes Hofscheier. Homogeneous spherical data of orbits in spherical embeddings. *Transform. Groups*, 20(1):83–98, 2015.

[GW09] Roe Goodman and Nolan R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[GZ05] Vincent Guedj and Ahmed Zeriahi. Intrinsic capacities on compact Kähler manifolds. *J. Geom. Anal.*, 15(4):607–639, 2005.

[Hoc65] Gerhard Hochschild. *The structure of Lie groups*. Holden-Day, Inc., San Francisco-London-Amsterdam, 1965.

[KKLV89] Friedrich Knop, Hanspeter Kraft, Domingo Luna, and Thierry Vust. Local properties of algebraic group actions. In *Algebraische Transformationsgruppen und Invariantentheorie*, volume 13 of *DMV Sem.*, pages 63–75. Birkhäuser, Basel, 1989.

[KKV89] Friedrich Knop, Hanspeter Kraft, and Thierry Vust. The Picard group of a G-variety. In *Algebraische Transformationsgruppen und Invariantentheorie*, volume 13 of *DMV Sem.*, pages 77–87. Birkhäuser, Basel, 1989.

[Kno91] Friedrich Knop. The Luna-Vust theory of spherical embeddings. In *Proceedings of the Hyderabad conference on algebraic groups held at the School of Mathematics and Computer/Information Sciences of the University of Hyderabad, India, December 1989*, pages 225–249. Madras: Manoj Prakashan, 1991.

[Kno94] Friedrich Knop. The asymptotic behavior of invariant collective motion. *Invent. Math.*, 116(1-3):309–328, 1994.

[Los09] Ivan V. Losev. Uniqueness property for spherical homogeneous spaces. *Duke Math. J.*, 147(2):315–343, 2009.

[LV83] Dominique Luna and Thierry Vust. Plongements d'espaces homogènes. *Comment. Math. Helv.*, 58:186–245, 1983.

[LZZ] Yan Li, Bin Zhou, and Xiaohua Zhu. K-energy on polarized compactifications of Lie groups. arxiv:1701.00306.

[Mai00] Vincent Maillot. Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables. *Mém. Soc. Math. Fr. (N.S.)*, (80):vi+129, 2000.

[Nyb] Thomas Nyberg. Constant Scalar Curvature of Toric Fibrations. PhD thesis, Columbia University.

[Pas08] Boris Pasquier. Variétés horosphériques de Fano. *Bull. Soc. Math. Fr.*, 136(2):195–225, 2008.

[Pas17] Boris Pasquier. A survey on the singularities of spherical varieties. *EMS Surv. Math. Sci.*, 4(1):1–19, 2017.

[Per14] Nicolas Perrin. On the geometry of spherical varieties. *Transform. Groups*, 19(1):171–223, 2014.

[Pez10] Guido Pezzini. Lectures on spherical and wonderful varieties. *Les cours du CIRM*, 1(1):33–53, 2010.

[PS10] Fabio Podestà and Andrea Spiro. Kähler-Ricci solitons on homogeneous toric bundles. *J. Reine Angew. Math.*, 642:109–127, 2010.

[Raz] Aleksis Raza. Scalar curvature and multiplicity-free actions. PhD thesis, Imperial College.

[Raz07] Aleksis Raza. An application of Guillemin-Abreu theory to a non-Abelian group action. *Differ. Geom. Appl.*, 25(3):266–276, 2007.

[Ric82] Roger Richardson. Orbits, invariants, and representations associated to involutions of reductive groups. *Invent. Math.*, 66(2):287–312, 1982.

[Ruz11] Alessandro Ruzzi. Smooth projective symmetric varieties with Picard number one. *Internat. J. Math.*, 22(2):145–177, 2011.

[Tim11] Dmitri A. Timashev. *Homogeneous spaces and equivariant embeddings*, volume 138 of *Encyclopaedia of Mathematical Sciences*. Springer, Heidelberg, 2011. Invariant Theory and Algebraic Transformation Groups, 8.

[vdB05] Erik P. van den Ban. The Plancherel theorem for a reductive symmetric space. In *Lie theory*, volume 230 of *Progr. Math.*, pages 1–97. Birkhäuser Boston, Boston, MA, 2005.

[Vus74] Thierry Vust. Opération de groupes réductifs dans un type de cônes presque homogènes. *Bull. Soc. Math. France*, 102:317–333, 1974.

[Vus90] Thierry Vust. Plongements d'espaces symétriques algébriques: Une classification. (Embeddings of algebraic symmetric spaces: a classification). *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, 17(2):165–195, 1990.

[ZZ08] Bin Zhou and Xiaohua Zhu. Relative K-stability and modified K-energy on toric manifolds. *Adv. Math.*, 219(4):1327–1362, 2008.

Thibaut Delcroix, Département de mathématiques et applications, École normale supérieure, CNRS, PSL Research University, 75005 Paris, France
E-mail address: thibaut.delcroix@ens.fr