Kähler-Einstein metrics and K-stability of Fano spherical varieties

Thibaut Delcroix

Institut Fourier (Grenoble, France)

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Summary

1. Introduction

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2. Spherical varieties
3. K-stability criterion
4. $K \times K$-invariant geometry of group compactifications

Introduction

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Theorem (Chen-Donaldson-Sun / Tian)

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Checking K-stablity involves checking the sign of some numerical invariant (the *Donaldson-Futaki invariant*) for all *test configurations* for (X, K_X^{-1}). Very difficult task to describe all test configurations, no general method to compute Donaldson-Futaki invariants.

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Remark

It is enough to check K-stability for **special** *test configurations, i.e. whose central fiber is normal.*

Equivariant K-stability

Assume further that a complex reductive group G acts on X.

Theorem (Datar-Székelyhidi)

X is Kähler-Einstein if and only if it is **equivariantly K-stable***,*
i.e. K-stable with respect to special, G-equivariant test configurations.

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Upshot: Need to consider fewer test configurations.

If X has enough automorphisms, may be able to compute K-stability.

Very symmetric manifolds

- Compact homogeneous manifold:
  $X = G/P = K/K \cap P$
  (G semisimple, P parabolic subgroup of G, K maximal compact subgroup of G)

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  Will consider a large class of almost homogeneous manifolds: the *spherical manifolds*.

Reductive groups

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- Examples: $(\mathbb{C}^*)^n$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{PSL}_n(\mathbb{C})$, more generally:

$$G = \frac{\prod(\text{simple complex Lie groups}) \times (\mathbb{C}^*)^n}{\text{finite central subgroup}}$$

Root system and representations

Let $T \simeq (\mathbb{C}^*)^r$ be a maximal torus of G.

$\mathfrak{X}(T) = \{\lambda : T \longrightarrow \mathbb{C}^* \text{ algebraic group morphisms}\} \simeq \mathbb{Z}^r$
group of (algebraic) *characters* of T.

$\Phi \subset \mathfrak{X}(T) \setminus \{0\}$ root system of (G, T) $\quad \supset \Phi^+$ positive roots.

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All finite dimensional representations of G are decomposable into a finite direct sum of irreducible representations. There is a bijection

$$\{\lambda \in \mathfrak{X}(T); \forall \alpha \in \Phi^+, \langle \alpha, \lambda \rangle \geq 0\} \longrightarrow \{\text{irreducible representations of } G\}$$
$$\lambda \longmapsto V_\lambda$$

V_λ is called the irreducible representation with *highest weight* λ.

Spherical varieties

Definition

G/H is called a **spherical homogeneous space** *if for any G-linearized line bundle L on G/H,*

$$H^0(G/H, L) = \bigoplus_{\lambda \in \mathfrak{X}(T), \langle \lambda, \Phi^+ \rangle \geq 0} V_\lambda^{m_\lambda}$$

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Definition

X G-almost homogeneous variety, with open orbit $G/H \subset X$ is called **spherical** *if G/H is spherical and X is normal.*

Horospherical varieties

Definition

A spherical homogeneous space G/H such that

- $N_G(H) := \{g \in G; gH = Hg\}$ *is a* **parabolic** *subgroup of* G *(i.e. closed connected and* $G/N_G(H)$ *compact)*
- *and* $N_G(H)/H$ *is a torus* $(\mathbb{C}^*)^r$,

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$\Rightarrow$ structure of homogeneous fibration in tori over a compact homogeneous manifold.

$X \supset G/H$ spherical is called *horospherical*.

The fibration structure may extend to X: *homogeneous toric bundles*, or not: *colored horospherical varieties*.

Group compactifications

X normal biequivariant *compactification* of a reductive group G, *i.e.* X compact, normal variety, $G \subset X$ Zariski open and $G \times G \curvearrowright X$ extending the natural action on G:

$$(g_1, g_2) \cdot g = g_1 g g_2^{-1}$$

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Then X is spherical under the action of $G \times G$.

Remark

Compact toric varieties may be considered as horospherical varieties or group embeddings.
Both classes are much larger than toric varieties.

Moment polytope

Let X be a spherical variety which is $\mathbb{Q}$-Fano (*i.e.* a multiple of the anticanonical (Weil) divisor is Cartier and ample)

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Definition

The **moment polytope** $\Delta^+ \subset \mathfrak{X}(T) \otimes \mathbb{R}$ *of* X *is*

$$\Delta^+ := \overline{\left\{ \frac{\lambda}{k} \in \mathfrak{X}(T) \otimes \mathbb{Q}; \quad m_\lambda = 1 \text{ in } H^0(X, K_X^{-k}), \ k >> 1 \right\}}$$

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$$\Phi_P^+ := \{\alpha \in \Phi^+; \ \exists p \in \Delta^+, \langle \alpha, p \rangle > 0\}$$

$$2\rho_P := \sum_{\alpha \in \Phi_P^+} \alpha$$

$$\mathrm{bar}_{DH}(\Delta^+) := \int_{\Delta^+} p \prod_{\alpha \in \Phi_P^+} \langle \alpha, p \rangle \, dp / \mathrm{Vol}_{DH}(\Delta^+)$$

Valuation cone

To a spherical variety $X \supset G/H$ can also be associated several cones depending only on the open orbit G/H.
A fundamental one is the *valuation cone*, which consists of the G-invariant, $\mathbb{Q}$-valued valuations on the field of meromorphic functions on G/H.

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We will use in the statement a cone $\Sigma \subset (\mathfrak{X}(T) \otimes \mathbb{R})^*$ which is "essentially" the dual cone of the valuation cone. This cone will be given more explicitly for horospherical varieties and group compactifications.

K-stability criterion

Main Theorem

X $\mathbb{Q}$-Fano spherical variety with moment polytope Δ^+.
X is equivariantly K-stable if and only if

$$\mathrm{bar}_{DH}(\Delta^+) \in 2\rho_P + \mathrm{Relint}(\Sigma).$$

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Combined with the work of Datar and Székelyhidi, this yields

Corollary

Assume X is smooth and Fano. If $\mathrm{bar}_{DH}(\Delta^+) \in 2\rho_P + \mathrm{Relint}(\Sigma)$, then X is Kähler-Einstein. If $\mathrm{bar}_{DH}(\Delta^+) \in 2\rho_P + \Sigma$ then X is K-semistable.

Horospherical manifolds

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Corollary

Let X be a smooth and Fano horospherical manifold. The following are equivalent:

- *X is K-stable,*
- *X is K-semistable,*
- *the Futaki invariant of X is zero,*
- $\mathrm{bar}_{DH}(\Delta^+) = 2\rho_P$.

Generalization of the case of toric manifolds (Wang-Zhu 2004) and the case of homogeneous toric bundles (Podesta-Spiro 2010).
In fact, any special equivariant test configuration for a Fano horospherical manifold is a product test configuration.

Theorem (Pasquier 2009)

There are infinite families of smooth and Fano horospherical varieties with Picard number one, which are not homogeneous under a larger group. Their automorphism group is not reductive.

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Conjecture (Odaka-Odaka 2013)

Fano manifolds with Picard number one are K-semistable.

Fujita (2015) obtained two counter examples.

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Fano manifolds with Picard number one are K-semistable.

Fujita (2015) obtained two counter examples.
They are not exceptional:

Corollary

The examples of Pasquier are not K-semistable.

Group compactifications

Let X be a compactification of the reductive group G.
All irreducible representations in $H^0(X, K_X^{-r})$ are of the form $V_\lambda \otimes V_\lambda^*$.
The moment polytope Δ^+ lives in the subspace

$$\{(\lambda, -\lambda); \lambda \in \mathfrak{X}(T) \otimes \mathbb{R}\} \subset \mathfrak{X}(T \times T) \otimes \mathbb{R} = (\mathfrak{X}(T) \times \mathfrak{X}(T)) \otimes \mathbb{R}$$

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Identify Δ^+ with its image $p_1(\Delta^+)$ under the first projection.
In fact all the data may be transferred to this space, to recover:

Theorem (D. 2015)

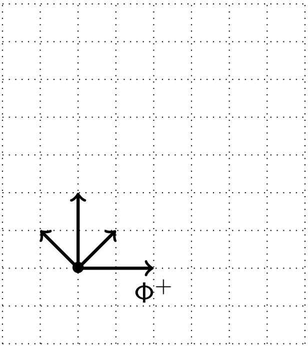
A smooth and Fano group compactification is Kähler-Einstein if and only if

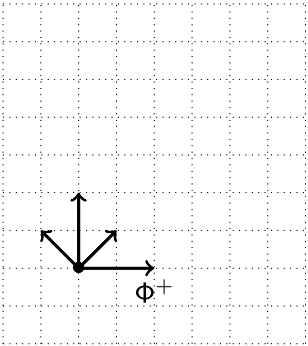
$$\mathrm{bar}_{DH}(\Delta^+) \in 2\rho + \mathrm{Relint}(\mathrm{Cone}(\Phi^+))$$

where $2\rho = \sum_{\alpha \in \Phi^+} \alpha$.

Examples

Moment polytopes of two smooth and Fano compactifications of $Sp_4(\mathbb{C})$, which are not homogeneous or toric under any group action.





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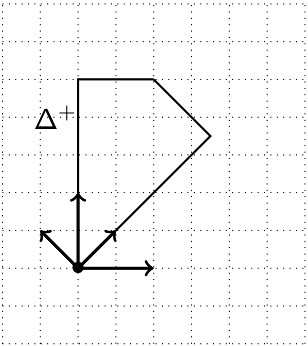

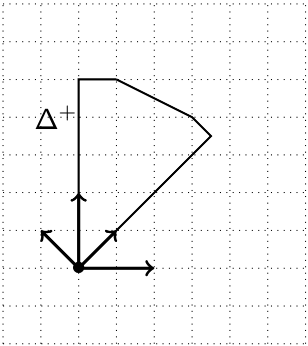

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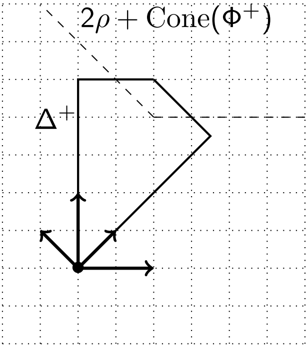

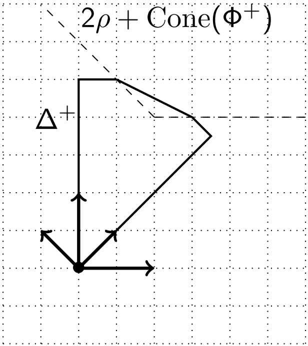

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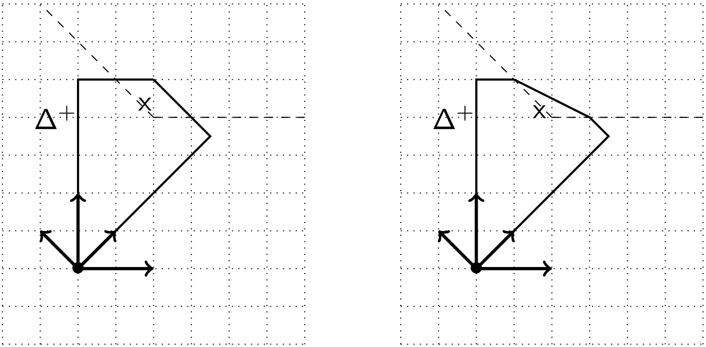

- The first one is Kähler-Einstein.
- The second one admits no Kähler-Ricci solitons.

KAK decomposition

Let $\mathfrak{a}^+$ be the positive Weyl chamber:

$$\mathfrak{a}^+ := \mathrm{Cone}(\Phi^+)^\vee$$

in

$$\mathfrak{a} := (\mathfrak{X}(T) \otimes \mathbb{R})^* \simeq \mathfrak{t} \cap i\mathfrak{k} \subset \mathfrak{g}$$

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Proposition (KAK decomposition)

The intersection of a $K \times K$-orbit in G and $\exp(\mathfrak{a}^+)$ is a single point.
The intersection of a $K \times K$-orbit in G and $\exp(\mathfrak{a})$ is an orbit of the Weyl group $W = N_G(T)/T$ of Φ.

Given $\phi : G \longrightarrow \mathbb{R}$ $K \times K$-invariant function, define

$$u : \mathfrak{a} \to \mathbb{R}, \quad x \mapsto \phi(\exp(x))$$

It is invariant under the Weyl group W.

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Proposition (Azad-Loeb 1992)

$$\{\psi : G \to \mathbb{R};\ K \times K - \text{invariant},\ \text{smooth},\ \text{strictly psh}\}$$
$$\updownarrow\ 1:1$$
$$\{u : \mathfrak{a} \to \mathbb{R};\ W - \text{invariant},\ \text{smooth},\ \text{strictly convex}\}$$

Kähler-Einstein equation

The Kähler-Einstein equation

$$\mathrm{Ric}(\omega) = \omega$$

for ω $K \times K$-invariant reads, on G,

$$\partial\bar{\partial}\psi = \partial\bar{\partial}\phi,$$

where ψ and ϕ are $K \times K$-invariant smooth and strictly psh functions on G, global potentials of $\mathrm{Ric}(\omega)$ and ω on G.

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Proposition

In terms of the convex function u associated to ϕ, the equation becomes

$$\det(D^2u(x)) \prod_{\alpha\in\Phi^+} \langle \alpha, \nabla u(x)\rangle^2 = e^{-u(x)} \prod_{\alpha\in\Phi^+} \sinh^2 \langle \alpha, x\rangle .$$

Continuity method equation

Similarly, let $\omega_{\mathrm{ref}} \in 2\pi c_1(X)$ be a $K \times K$-invariant reference metric, and $t \in [0,1]$,

$$(*)_t \qquad \mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega_{\mathrm{ref}}$$

becomes

$$\det(D^2 u_t(x)) \prod_{\alpha\in\Phi^+} \langle \alpha, \nabla u_t(x)\rangle^2 = e^{-(tu_t(x)+(1-t)u_{\mathrm{ref}})} \prod_{\alpha\in\Phi^+} \sinh^2 \langle \alpha, x\rangle \, .$$

Remark

$(*)_1$ *is the Kähler-Einstein equation,*
$(*)_0$ *has solutions by the Calabi-Yau theorem.*

Greatest Ricci lower bound

Székelyhidi:

$$\begin{aligned} R(X) &:= \sup\{t \in [0,1]; \exists \omega \in 2\pi c_1(X), \mathrm{Ric}(\omega) \geq t\omega\} \\ &= \sup\{t, (*)_t \text{ has a solution}\} \end{aligned}$$

Li: $R(X) = 1$ if and only if X is K-semistable.

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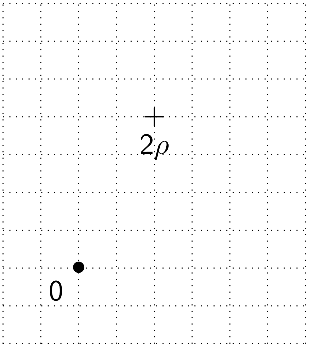
Theorem

If X is a group compactification, then $R(X)$ is equal to

$$\sup\{t \in [0,1]; 2\rho + \frac{t}{1-t}(2\rho - \mathrm{bar}_{DH}(\Delta^+)) \in \Delta^+ + (-\mathrm{Cone}(\Phi^+))\}.$$

Example



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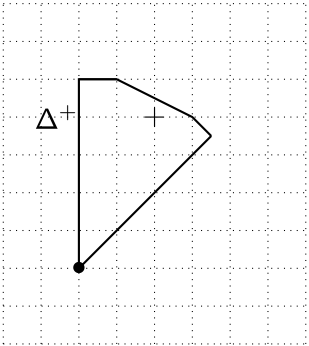

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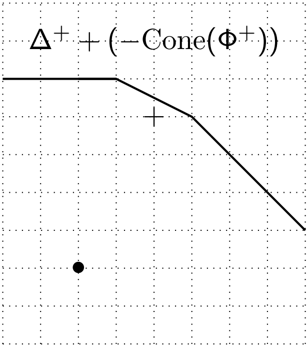

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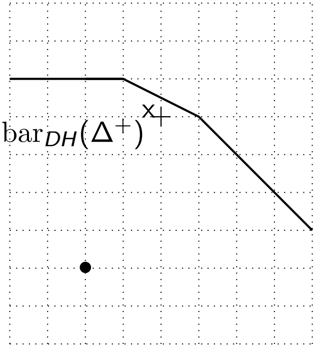

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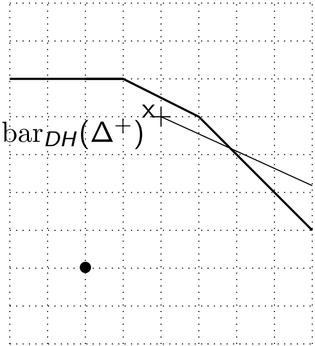

$$R(X) = \frac{1046175339}{1236719713} < 1$$

Key argument

Let

$$\nu_t := tu_t + (1-t)u_{\mathrm{ref}} + j$$

where

$$j(x) := -\ln\Big(\prod_{\alpha\in\Phi^+} \sinh^2\langle\alpha, x\rangle\Big)$$

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$$\nu_t := tu_t + (1-t)u_{\mathrm{ref}} + j$$

where

$$j(x) := -\ln\Big(\prod_{\alpha\in\Phi^+} \sinh^2\langle\alpha, x\rangle\Big)$$

Lemma

$$\forall \xi \in \mathfrak{a}^+, \qquad \int_{\mathfrak{a}^+} \langle \xi, \nabla \nu_t\rangle\, e^{-\nu_t} = 0$$

Upper bound

Let

$$v_{\Delta^+}(\xi) := \sup\{\langle p, \xi\rangle, p \in \Delta^+\}$$

Then

$$0 = t\int \langle \xi, \nabla u_t\rangle e^{-\nu_t} + (1-t)\int \langle \xi, \nabla u_{\mathrm{ref}}\rangle e^{-\nu_t} + \int \langle \xi, \nabla j\rangle e^{-\nu_t}$$
$$\leq \frac{t\,\langle \mathrm{bar}_{DH}(\Delta^+), \xi\rangle + (1-t)v_{\Delta^+}(\xi) + \langle -2\rho, \xi\rangle}{\mathrm{Vol}_{DH}(\Delta^+)}$$

Obtain

$$\forall \xi \in \mathfrak{a}^+, \quad \left\langle 2\rho + \frac{t}{1-t}(2\rho - \mathrm{bar}_{DH}(\Delta^+)), \xi \right\rangle \leq v_{\Delta^+}(\xi)$$

$$\Updownarrow$$

$$2\rho + \frac{t}{1-t}(2\rho - \mathrm{bar}_{DH}(\Delta^+)) \in \Delta^+ + (-\mathrm{Cone}(\Phi^+))$$

Lower bound

If X has no Kähler-Einstein metrics, then

$$0 = \int_{\mathfrak{a}^+} \langle \xi_t, \nabla \nu_t \rangle e^{-\nu_t} \to \frac{R(X)\langle \mathrm{bar}_{DH}(\Delta^+), \xi_\infty \rangle + (1 - R(X)) v_{\Delta^+}(\xi_\infty) + \langle -4\rho, \xi_\infty \rangle}{2\mathrm{Vol}_{DH}(\Delta^+)}$$

for some choice of ξ_t, as $t \to R(X)$.

$$\left\langle 2\rho + \frac{R(X)}{1 - R(X)}(2\rho - \mathrm{bar}_{DH}(\Delta^+)), \xi_\infty \right\rangle = v_{\Delta^+}(\xi_\infty)$$

Thank you!