Cohomology of wonderful models

§1 Recollections

V fin dim$/\mathbb{C}$.

Hyperplane arrangement $\rightsquigarrow$ matroid

$\mathcal{H} = \{ H_i \subseteq \mathbb{P}(V) \}$

$M_{\mathcal{H}}$ representable

assume $\bigcap_{\text{all}} H_i = \emptyset$

$\alpha_i \in V^\vee$

Intersection lattice

$P(\mathcal{H}) = \left\{ \begin{array}{l} \text{intersections} \quad \text{"}\leq\text{"} \\ X = H_1 \cap \dots \cap H_n, \ \text{"}\supseteq\text{"} \end{array} \right\}$

Lattice of flats $P(M)$

Properties:

1) $X \vee Y = X \cap Y$, $0 = \mathbb{P}(V)$.

2) $X \wedge Y = \bigcap_{Y \supseteq H \leq X} H$, $1 = \emptyset$

} lattice.

geometric lattice.

3) Atoms $= \mathcal{H}$.

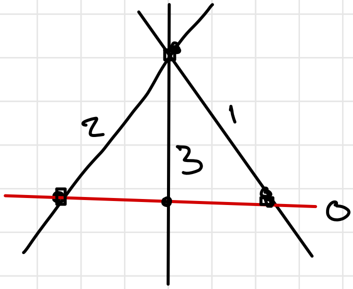
} atomistic $X = \bigvee_{H \leq X} H$.

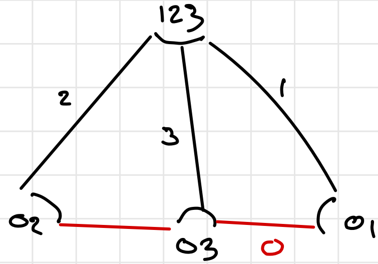
4) $rk : P(\mathcal{H}) \to \mathbb{Z}_{\geq 0}$ codim

} semimodular

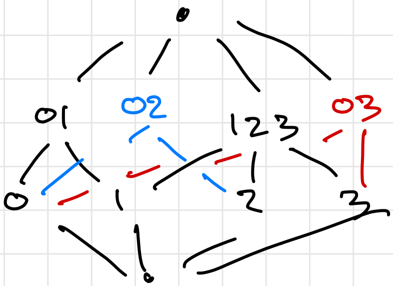
$rk(X) + rk(Y) \geq rk(X \vee Y) + rk(X \wedge Y)$.

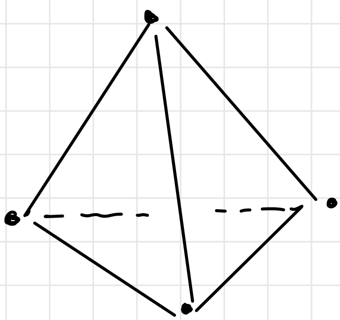
Examples :

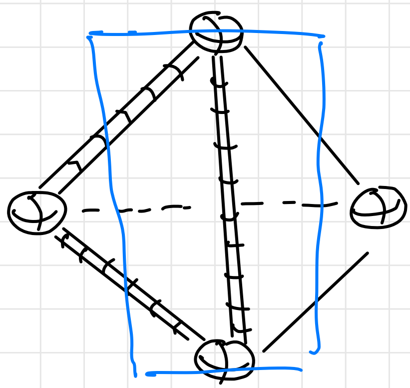




P(ℋ)







$P(\mathcal{H}) = 2^{\{0,1,2,3\}}$.

§1.2 Wonderful models

→ Fix $K \subseteq P(\mathcal{F})^{>0}$ upwards closed, nonempty. (ultimately: $K_{max} = P(\mathcal{F})^{>0}$).

$\cup$ lines in X.

$$\mathbb{P}(V) \setminus \bigcup_{\text{all}} H_i \xhookrightarrow{\text{open}} \underset{\text{compact.}}{Y_K} \xhookrightarrow{\text{closed}} \prod_{X \in K} \mathbb{P}(V/\tilde{X})$$

→ $\underline{\text{Ex } 0}$: $K = \{1\} = \{\emptyset\}$: $Y_{\{1\}} = \mathbb{P}(V)$

→ Other Y_K obtained inductively by blowups, going down in K (blow up low dim intersections first).

→ $\mathbb{P}(V) \setminus \bigcup H_i \xhookrightarrow{\text{open}} Y_K$

complement normal crossings divisor

irreducible components : D_X $\quad X \in K \setminus 1$

$D_{X_1} \cap \dots \cap D_{X_n} \neq \emptyset \iff$ up to Σ_n:
$X_1 \leq \dots \leq X_n$.

§2 Cohomology of Y_K: first remarks.

$\rightsquigarrow$ classes $D_X \in H^2(Y_K)$ $\forall X \in K \setminus 1$.

convention: $D_1 \in H^2(Y_K)$ "generic hyperplane"

$$\pi : Y_K \longrightarrow \mathbb{P}(V)$$

$$D_1 = -\pi^{-1}\left(\begin{matrix}\text{hyperplane}\\ \text{class in } \mathbb{P}(V)\end{matrix}\right).$$

Relations: 1) $D_X \cdot D_Y = 0$ if X, Y incomparable.

If:

2) $H \in K$: $\pi^{-1}(H) = \sum_{H \leq X \in K \setminus 1} D_X = -D_1$

$$\rightsquigarrow \sum_{H \leq X} D_X = 0$$

⌜ More generally: take $X \in K$, then

$$\pi_X : Y_K \longrightarrow \mathbb{P}(V/\hat{X}).$$

dim $\mathrm{rk}(X) - 1$

$$\sum_{X \leq Y} D_Y = \pi_X^{-1}(G) \qquad \begin{matrix} \cup\!\!\mid & & \cup\!\!\mid \\ \end{matrix} \quad G \text{ hyperplane.}$$

$$\rightsquigarrow \left(\sum_{X \leq Y} D_Y\right)^{\mathrm{rk}(X)} = 0$$

⌟

<u>Theorem</u> (de Concini–Procesi)

$\mathcal{H}$ hyperplane arrangement, $K_{max} = P(\mathcal{H})^{>0}$,

then there is an Iso:

$$\boxed{A^*(M_{\mathcal{H}}) =} \frac{\mathbb{Z}[D_X : X \in K]}{\left(\begin{array}{ll} D_X \cdot D_Y & X, Y \text{ incomp.} \\ \sum\limits_{H \leq X} D_X & : H \text{ hyperplane} \end{array}\right)} \xrightarrow{\cong} H^{2*}(Y_{K_{max}}).$$

$\cong$

§3 <u>Algebraic intermezzo</u>

$M = (E, I)$ matroid

$P(M)$ lattice of flats.

<u>Def</u>: The Chow ring of M is the graded ring

$$A^*(M) = \frac{\mathbb{Z}[D_X : X \in P(M)^{>0}]}{\left(\begin{array}{ll} D_X \cdot D_Y & : X, Y \text{ incomp} \\ \underset{H \leq X}{\underset{e \in X}{\sum}} D_X & : \underset{H \text{ atom}}{e \in E} \end{array}\right)}$$

Variant: for $K \subseteq P(M)^{>0}$ upwards closed, define

$$A^*(K) = \frac{\mathbb{Z}[D_X : X \in K]}{\begin{pmatrix} ① \ D_X \cdot D_Y & : X, Y \text{ incomp} \\ ② \left(\sum_{Y \leq Z} D_Z\right)^{rk(Y)} & : \\ ③ \ D_X \left(\sum_{Y \leq Z} D_Z\right)^{rk(Y)-rk(X)} & : X < Y \end{pmatrix}}$$

Thm (Feichtner – Yuzvinsky): $A^*(M) = A^*(P(M)^{>0})$.

Idea of proof: have to check: $D_X \left(\sum_{Y \leq Z} D_Z\right)^{rk(Y)-rk(X)}$

are already in $\begin{pmatrix} D_X \cdot D_Y \\ \left(\sum_{H \leq Z} D_Z\right) & H \text{ atom} \end{pmatrix}$.

idea: induction on $d = rk(Y) - rk(X)$.

Thm: For $K \subseteq P(\mathcal{H})^{>0}$ upwards closed:

$$A^*(K) \longrightarrow H^{2*}(Y_K).$$

Proof by induction: start $K = \{i\}$, $Y_K = \mathbb{P}(V)$.

For inductive step, recall:

$X_0 \in K$ minimal $\rightsquigarrow$ $K \setminus X_0$

$$K/X_0 = \left\{ \begin{matrix} \text{proper intersections} \\ \text{of } H_i\text{'s in } X_0 = \mathbb{P}^{\dim(X_0)} \end{matrix} \right\} \cong \left\{ X \in K : X_0 < X \right\}$$

different rank: $rk(X) - rk(X_0)$

Blow-up:

$$\begin{tikzcd} \mathbb{P}(\nu) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y_K \arrow[d] \\ Y_{K/X_0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y_{K \setminus X_0} \arrow[d] \\ \mathbb{P}^{\dim X_0} = X_0 \arrow[r] & \mathbb{P}(V) \end{tikzcd}$$

Lemma (Keel)

$$\begin{tikzcd} \mathbb{P}(\nu) \arrow[r] \arrow[d] & Bl_Z(W) \arrow[d] \\ Z \arrow[r, hook] & W \end{tikzcd}$$

blowup.

If $H^*(W) \to H^*(Z)$ surjective with kernel J, then

$$H^*(Bl_Z(W)) = H^*(W)[E] \Big/ \left(J \cdot E, \ \tilde{ch}_{-E}(\nu) \right)$$

exceptional divisor

$\tilde{ch}_t(\nu)$ lift of Chern polynomial of normal bdl ν?

$$\left(\begin{array}{l} ① \; D_X \cdot D_Y \\ ② \; \left(\sum_{Y\le Z} D_Z\right)^{rk(Y)-rk(X_0)} \\ ③ \end{array}\right) \qquad \left(\begin{array}{c} ① \\ ② \\ ③ \end{array}\right) \qquad \left(\begin{array}{l} ①, ②, ③ \\ \cdot \; D_X : X_0 \not< X \\ \cdot \end{array}\right).$$

$$\begin{tikzcd} \mathbb{Z}[D_X : X_0 < X] \arrow[d, two heads] & & \mathbb{Z}[D_X : X \in K \setminus X_0] \arrow[ll, two heads] \arrow[d, two heads] \arrow[dll] \\ H^*(Y_{K/X_0}) & & H^*(Y_{K\setminus X_0}) \arrow[ll, two heads] \end{tikzcd}$$

$$\left.\begin{array}{c} D_X \\ 0 \end{array}\right\} \longleftarrow\!\shortmid D_X$$

Normal bundle of $Y_{K/X_0} \hookrightarrow Y_{K\setminus X_0}$

① X_0 hyperplane: line bundle L, corresponding to divisor $\sum_{X_0<X} D_X$.

② X_0 transverse $\cap$ of hyperplanes $\rightsquigarrow L^{\oplus rk(X_0)}$.

By Keel:

$$H^*(Y_K) = H^*(Y_{K\setminus X_0})[D_{X_0}] \Big/ \begin{array}{l} \cdot \; D_{X_0} \cdot \} \\ \cdot \; \tilde{c}h_{-D_{X_0}}(\nu) \end{array}$$

$$\left(D_{X_0} + \sum_{X_0<X} D_X\right)^{rk(X)}$$

② w/ X_0

$$= \mathbb{Z}[D_X : X \in K] \Big/ \begin{array}{l} \cdot \; ①, ②, ③ \text{ w.o/ } X_0, \\ \cdot \left\{ \begin{array}{ll} D_{X_0} \cdot D_X & : X_0 \not< X \\ D_{X_0} \left(\sum_{X<Y} D_Y\right)^{rk(X)-rk(X_0)} & \end{array}\right. \end{array}$$

① w/ X_0

③ w X_0