Semisimple Principal Toric fibrations

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Talk goals:

- Construction of a principal fibration from a principal bundle. Examples
- Compatible Kähler metrics. Embedding of potentials. An integration formula.

Hamiltonian actions

Let (M^{2n}, ω) be a compact connected symplectic manifold with a smooth G-action.

Definition

The action of G is hamiltonian if there exists a map:

$$\mu_\omega : M \to \mathfrak{g}^*$$

such that :

- For each $\xi \in \mathfrak{g}$, every component of μ_ω along ξ is hamiltonian, i.e. $d\langle \mu_\omega, \xi \rangle = i_\xi \omega$.

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Guiding example throughout the talk :

Example

$(X, \omega, \mathbb{S}^1) = (\mathbb{P}^1, \omega_{FS}, \mathbb{S}^1)$. $\mu_{\omega_{FS}}(\mathbb{P}^1) = [-1/2, 0]$.

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- The principal $\mathbb{S}^1$-bundle $\mathcal{U}(1)$ over $\mathbb{P}^1$, constructed from $\mathcal{O}(-1)\setminus\{0\}$, by the choice of a hermitian metrics on $\mathcal{O}(-1)$ (*the Hopf fibration*).

Principal fibration

Definition

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- Other Hirzeburch surfaces : $(\mathcal{U}(k) \times \mathbb{P}^1)/\mathbb{S}^1 \simeq \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k))$. Here $\mathcal{U}(k)$ is the circle bundle of $\mathcal{O}(-k)$.

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Definition

A connection on P is the choice of a right invariant horizontal distribution $H \subset TP$ s.t. $H_p \oplus G_p = T_p P$. Equivalently, it's the choice of an equivariant $\theta \in \Omega^1(P, \mathfrak{g})$ s.t. $\theta(A_*) = A$ with $\ker \theta := H$.

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Fact

Every principal bundle admits a connection.

A connection θ on P induces a connection θ on $X \times P$, hence a horizontal distribution on $X \times P$, identified TY, $Y = (X \times P)/G$:

$$TY \simeq TX \oplus TB$$

Semi-simple principal fibrations

Definition

Given the data

- $(X, \omega, \mathbb{T}^r_X)$ compact Kähler $2m$-fold with isometric hamitonian action. $\mathbb{T} \simeq \mathfrak{t}/2\pi\Lambda$, Λ a lattice with basis $\{\xi_i, \ldots, \xi_r\}$.

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we obtain the principal fibration $Y := (X \times P)/\mathbb{T}_{X \times P}$ with a complex structure given by $(TY, J_Y) \simeq (TX \oplus TB, J_X \oplus J_B)$ and $\theta = 0$ on $TX \oplus TB$. . Action on $X \times P$: $t(x, p) := (tx, pt^{-1})$.

Remark

- Can work with $B = B_1 \times \cdots \times B_k$ (B_i Hodge cscK). But to simplify, $k = 1$.

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- $\mathbb{T}$-principal bundle over B with curvature $d\theta \iff$ Fiberwise product of r principal $\mathbb{S}^1$-bundle $\mathcal{U}_i(1)$ associated to r hermitian line bundles L_i over B with $2\pi c_1(L_i) = p_i\pi^*\omega_B$, where $p = \sum_{i=1}^r p_i\xi_i$.

Bundle compatible metrics

Example

$(\mathbb{P}^1, \omega_{FS}, \mathbb{S}^1)$, $(B, \omega_B) = (\mathbb{P}^1, \omega_{FS})$ with principal bundle $\mathcal{U}(1)$, and Y is the Hirzebruch complex surface $\mathbb{P}^1(\mathcal{O} \oplus \mathcal{O}(-1))$.

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A bundle-compatible Kähler metric on Y is a basic 2-forms $\widetilde{\omega}$ on $X \times P$ given by: $\widetilde{\omega} = \omega_h + \omega_v$ where

- $\omega_h := \omega + (\langle p, \mu_\omega \rangle + c)\pi_B^* \omega_B$ is the horizontal part. Here, $c \in \mathbb{R}$ is such that $\langle p, \mu_\omega \rangle + c > 0$ on the moment polytope.

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Equivalently :

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Positive definite + basic + closed $\Rightarrow$ $\widetilde{\omega}$ descends to a Kähler form on Y.

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The embedding $C^\infty_{\mathbb{T}}(X) \hookrightarrow C^\infty_{\mathbb{T}}(Y)$ induces a well-defined embedding of Kähler potentials $K_{\mathbb{T}}(X, \omega) \hookrightarrow K_{\mathbb{T}}(Y, \widetilde{\omega})$ In other words, if $\omega_\phi = \omega + d_X d^c_X \phi$ then $\widetilde{\omega}_\phi = \widetilde{\omega} + d_Y d^c_Y \phi$, i.e. the map $[\omega] \to [\widetilde{\omega}]$ is well-defined.

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Sketch of proof : Let μ_ϕ be the moment map of ω_ϕ. By the equality :

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we see that $\widetilde{\omega}_\phi = \widetilde{\omega} + d_Y d^c_Y\phi$ is equivalent to

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Both sides vanish on vertical vectors (vectors generated by local generator of $\mathfrak{t}_{X,P}$) and equal on horizontal vectors . □

An integration formula

By compatibility, on the horizontal part of $X \times P$:

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If $B = B_1 \times \cdots \times B_k$, then $p = \prod_{i=1}^k (\langle p_i, \mu_\phi \rangle + c_i)$. Hence for all $f \in C^\infty_{\mathbb{T}}(X) \hookrightarrow C^\infty_{\mathbb{T}}(Y)$:

$$\int_Y f \widetilde{\omega}_\phi^{[n+b]} = \mathrm{vol}(B, \omega_B) \int_X p(\mu_\phi) f \omega_\phi^{[n]}$$

Allows construction of the weighted Monge-Ampère measure !