Weighted scalar curvature and extremal problem on semi-simple principale fibratio.

- (X, ω_X, J_X) compact Kähler manifold.

  $T := \mathfrak{t}/\Lambda 2\pi$, $T_X \hookrightarrow Aut_r(X)$, Δ moment polytope

- $\pi_B : P \longrightarrow \prod_{a=1}^{k} (B_a, \omega_a) = (B_a, \omega_B)$

  principale T-bundle. We suppose that each ω_a are cscK and $\omega_a \in H^2(B_a, \mathbb{Z})$.

We suppose that

$$2\pi c_1(P) = \sum_{a=1}^{k} \pi_B^*[\omega_a] \otimes p_a \qquad p_a \in \mathfrak{t}$$

We choose a connection θ s.t.

$$d\theta = \sum_{a=1}^{k} \omega_a \otimes p_a$$

- $Y := \dfrac{P \times X}{T} \circlearrowleft T_Y \subset \mathrm{Aut}_\pi(Y)$

$$J_Y := J_B \oplus J_X$$

on $Z := P \times X$

$$\omega_Y := \omega_X + \sum_{a=1}^{k} \overbrace{\left(\langle \mu, p_a \rangle + c_a\right)}^{>0} \omega_a + \langle d\mu, \theta \rangle$$

μ is the Δ-normalized moment map

ω_Y is T_Y-invariant.

Lemma: $v \in \mathcal{C}^\infty(\Delta, \mathbb{R}_{>0})$, ω_Y and ω_X as above. Then

$$\mathrm{Scal}_v(\omega_Y) = \frac{1}{p(\mu)} \times \mathrm{Scal}_{pv}(\omega_X) + v(\mu) q(\mu)$$

where:

- $p(\mu) := \prod_{a=1}^{k} (\langle p_a, \mu \rangle + c_a) > 0$
- $q(\mu) = \sum_{a=1}^{k} \frac{s_a}{\langle p_a, \mu \rangle + c_a}$

$s_a = \mathrm{Scal}(\omega_a) \in \mathbb{R}$

$$\mathrm{Scal}_v(\omega) = v(\mu_\omega)\,\mathrm{Scal}(\omega) + \Delta_\omega v(\mu_\omega) + \sum_{1 \le i,j \le r} v_{,ij}\, g(\xi_i^*, \xi_j^*)$$

$(\xi_1, \dots, \xi_r)$ basis of t^*

In particular, ω_Y is (v, w)-cscK iff ω_X is $(p v, \tilde{w})$-cscK with

$$\tilde{w}(\mu) = p(\mu)\big(w(\mu) - v(\mu)\, q(\mu)\big)$$

ω_Y is (v, w)-cscK:

$$\mathrm{Scal}_v(\omega_Y) = w$$

<u>Sketch of proof:</u>

X^0 be the dense open subset of X of regular orbit

$$TX^0 = H \overset{\perp}{\oplus} F_X \overset{\perp \omega_X}{\oplus} J_X F_X \qquad \{\xi_1, \ldots, \xi_r\} \text{ basis of } F$$

$$\omega_X = \omega_H + \sum_{1 \leq i,j < r} H_{ji}\, \eta_i \wedge J_X \eta_j$$

where $H_{ij} := g_{\omega_X}(\xi_i^X, \xi_j^X)$ $\qquad \eta_i|_H = 0$, $\quad \eta_i(\xi_j^X) = \delta_{ij}$

$\eta_i(\overline{J}_X \xi_j) = 0$

Locally, the Ricci form ρ_X of ω_X is given

$$\rho_X = -\frac{1}{2} dd^c \log\left(\frac{\omega_X^{[m]}}{Vol_X}\right)$$

$\omega^{[k]} = \frac{\omega^k}{k!}$, $1 \le k \le m$, where Vol_X is the volume form of the flat Kähler metric in any holomorphic system of coordinates.

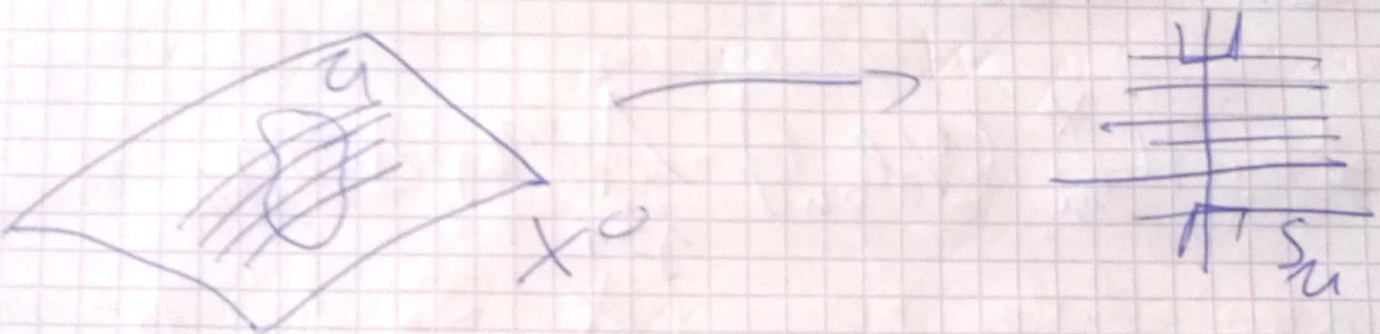
- $TX^0 = H \oplus \underbrace{\xi_X \oplus \overline{J}_X \xi_X}_{\xi_{\mathbb{C}}}$

$[\xi_{\mathbb{C}}, \xi_{\mathbb{C}}] = \{0\} \subset \xi_{\mathbb{C}}$

By Frobenius theorem, $\exists\, \pi : U \subset X^0 \longrightarrow S_U \subset \mathbb{C}^{m-2}$ an holomorphic map.

S_U is the space of sheaf.

Let Vol_{S_U} be the ~~f~~ volume form of the



flat Kähler metric on S_x in any system of coordinates.

sec° of $\Lambda^{m-n} H^*$

$$Q^{-1}\,\omega_H^{[m-n]} = \pi_{S_U}^{*}\, Vol_{S_U}$$

Q smooth fc° on U. sec° $\Lambda^{m-n} H^*$

• $\eta_j + iJ\eta_j \in \Omega^{1,0}(\mathcal{U})$ holomorphic

Then we can choose

$$\mathrm{Vol}_X = \pi_{S_\mathcal{U}}^*(\mathrm{Vol}_{S_\mathcal{U}}) \bigwedge_{j=1}^{r} \eta_j \wedge J\eta_j$$

Then

$$\rho_X = -\frac{1}{2} dd^c \log\left(\frac{\omega_X^{[m]}}{\mathrm{Vol}_X}\right)$$

$$= -\frac{1}{2} dd^c \log\left(\frac{Q\, \pi_{S_\mathcal{U}}^*(\mathrm{Vol}_{S_\mathcal{U}}) \det(H) \bigwedge_{j=1}^{r} \eta_j \wedge J\eta_j}{\pi_{S_\mathcal{U}}^*(\mathrm{Vol}_{S_\mathcal{U}}) \bigwedge_{j=1}^{r} \eta_j \wedge J\eta_j}\right)$$

$$= dd^c\left[\underbrace{\frac{\log(\alpha) + \log\det H}{-2}}_{K_X}\right]$$

- (Y, ω_Y, J_Y)

$$\rho_Y \overset{\text{local}}{=} dd^c K_Y = -\frac{1}{2} dd^c \log\left(\frac{\omega_Y^{[m+n]}}{Vol_Y}\right)$$

$$\omega_Y = \omega_X + \sum_{a=1}^{k} \left(\langle p_a, \mu\rangle + c_a\right)\omega_a + \langle d\mu, \theta\rangle$$

By similar arguments:

$$K_Y = \sum_{a=1}^{k} K_a + K_X - \frac{1}{2}\log(p(\mu))$$

$\rho_a = dd^c(K_a)$, ρ_a is the Ricci form of ω_a

$$\mathrm{Scal}(\omega_Y) = \frac{* \, dd^c k_Y \wedge \omega_Y^{[m+n-1]}}{\omega_Y^{[m+n-1]}}$$

Weighted Mabuchi functionnal:

Let ω_0 be a Kähler form on (X, J_X), T_X-invariant, and $\tilde{\omega}_0$ its corresponding compatible form on (Y, J_Y)

- $$K_T(X, \omega_0) \longleftrightarrow K_T(Y, \tilde{\omega}_0)$$

$\tilde{\omega}_\varphi$ the Kähler metric on Y corresponding to ω_φ on X

. ~~M~~ $v \in \mathcal{C}^\infty(\Delta, \mathbb{R}_{>0})$

$w \in \mathcal{C}^\infty(\Delta)$

$$\mathcal{M}^Y_{v,w} : \mathcal{K}_T(Y, \tilde{\omega}_0) \longrightarrow \mathbb{R}$$

$$\left| \begin{array}{l} d_\varphi \mathcal{M}^Y(\dot{\varphi}) = - \displaystyle\int_Y \left(\mathrm{Scal}_v(\tilde{\omega}_\varphi) - w(\mu_\varphi)\right) \dot{\varphi}\, \tilde{\omega}_\varphi^{[m+n]} \\ \mathcal{M}^Y_{v,w}(0) = 0 \end{array} \right.$$

μ_φ is the Δ- normalized moment map of $\tilde{\omega}_\varphi$

<u>Corollary</u>: The restiction of $\mathcal{M}^Y_{v,w}$ to $\mathcal{K}_X(X, \omega_0) \hookrightarrow \mathcal{K}_{T}(Y, \tilde{\omega}_0)$ is

$$\mathcal{M}^X_{pv, \tilde{w}} \times \mathrm{Vol}(B, [\omega_B])$$

Calabi Problem on semi-simple principal fibration.

- (X, J) compact complex manifold
- α a Kähler class
- $\omega \in \alpha$ is extremal if

$$\mathcal{L}_{\omega^{-1}(d\,Scal(\omega))} J = 0$$

- By a Thm of Calabi, if there exists an extremal metric in α, $\exists\ \tilde{\omega} \in \alpha$ extremal K-invariant $K \underset{max}{\subset} Aut_r(X)$ torus.

• $\forall\, \omega \in \alpha$ Kähler, K-invariant

$$P_\omega^K := \{ f \in \mathcal{C}^\infty_K(X) \mid \mathcal{L}_{\omega^{-1}(df)} J = 0 \}$$

hamiltonian Killing potential

• $\Pi_\omega^K : L^2(X) \longrightarrow P_\omega^K$

L^2-$\perp$ projection.

• $\omega \in \alpha$ K-invariant is extremal

iff $\mathrm{Scal}(\omega) = \Pi_\omega^K(\mathrm{Scal}(\omega))$

• Futaki and Mabuchi shown that

$$\xi_{ext} := \omega^{-1}(d\,\Pi_\omega^K(\mathrm{Scal}(\omega)))$$

is independent of $\omega \in \alpha$, K-invariant.

- ~~Denote~~ Denote by μ_ω the moment map Δ-normalized of ω

$$\Pi_\omega^K(Scal(\omega)) = \underbrace{\langle \xi_{ext}, \mu_\omega \rangle + c_{ext}}_{\substack{\parallel \\ l_{ext}(\mu_\omega)}}$$

$l_{ext}(\mu_\omega)$ is the affine extremal function.

- $\omega \in \alpha$ K-invariant is extremal iff $Scal(\omega) = \underline{l_{ext}(\mu_\omega)}$.

Come back to fibration:

Let $T_Y \subset K_Y \underset{maxi}{\subset} Aut_1(Y)$

- $\{1\} \hookrightarrow T_Y \hookrightarrow K_Y$
- $\{0\} \longrightarrow t_Y \longrightarrow k_Y$
- $k_Y^* \longrightarrow t_Y^* \longrightarrow \{0\}$
- $\{0\} \longrightarrow Aff(t_Y^*) \hookrightarrow Aff(k_Y^*)$

<u>Lemma</u>: (X, ω_X, J_X), (Y, ω_Y, J_X)

- $T_X \underset{max}{\subset} Aut_1(X)$

- ω_B, Let $K_B \underset{max}{\subset} Aut_1(B)$ s.t. ω_B is K_B-invariant.

Then, ω_Y is invariant by a maximal torus $T_Y \subset K_Y \underset{max}{\subset} Aut_1(Y)$ s.t.

$$\{0\} \longrightarrow t_Y \longrightarrow k_Y \longrightarrow k_B \longrightarrow \{0\}$$

$k_Y := Lie(K_Y)$. Moreover ℓ_{ext} is in the image

$k_B := Lie(K_B)$ of $Aff(t_Y^*) \hookrightarrow Aff(k_Y^*)$

Sketch :

- Let $f \in P_{\omega_B}^{K_B}$, $f = \sum_{a=1}^{k} f_a$ with f_a hamiltonian Killing potential on (B_a, ω_a).
Let K the corresponding hamiltonian Killing vector field.

- $\tilde{K}$ the lift of K to P (via θ)

$$\hat{K} := \tilde{K} + \sum_{a=1}^{k} f_a \, \xi_{p_a}^{P}$$

$Z = X \times P \longrightarrow Y$ is a principal T_Z-bundle

$TZ = H \oplus t_Z$ w.r.t. θ.

1) $[\hat{K}, T_Z] = \{0\}$ 3) $[\hat{K}, H] \subset H$

2) $\mathcal{L}_{\hat{K}} (J_B \oplus J_X) = 0$

4) $\omega_Y = \sum_{a=1}^{k} (\langle p_a, \mu\rangle + c_a)\, \omega_a + \omega_X + \langle d\mu, \theta\rangle$

$= \sum_{a=1}^{k} c_a\, \omega_a + \omega_X + d\langle \mu, \theta\rangle$

$\omega_Y(\widehat{K}, \cdot) = -d\left(\sum_{a=1}^{k} f_a \left(\langle p_a, \mu\rangle + c_a\right)\right)$

$\widehat{k_B} \oplus t_X \subset TZ$ descends to an algebra of hamiltonian Killing vector fields on Y

$k_Y : \widehat{k_B} \oplus t_X$