# Weighted cscK working group Apr. 15 2022.

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$$\mathcal{E}^1_{\mathbb{T}}(X, \omega_0) := \left\{ \phi \in \mathrm{PSH}_{\mathbb{T}}(X, \omega_0),\ \int_X |\phi| (\omega_\phi)^m < \infty \right\}.$$

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**Today,** we will look into the extension of $\mathrm{M}_{v,w}$ to the space $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$.

By Lahdili, one has a Chen-Tian-like decomposition of $\mathrm{M}_{v,w}$ on $\mathcal{K}_{\mathbb{T}}$:

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1. Extend the entropy term. Main idea: in the case of polynomial $v$, extension of the weighted entropy on $X$ corresponds to extension of the unweighted entropy on an associated fibration.

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Precise statement:

**Theorem**

*The Chen-Tian formula gives the largest $d_1$-lsc extension of $\mathrm{M}_{v,w}$ on $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$. Furthermore, this extended $\mathrm{M}_{v,w}$ is linear in $v$, $w$, uniformly continuous in $w$ wrt $C^0(\Delta)$ and continuous in $v$ wrt $C^1(\Delta)$*

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$$\mathrm{Ent}(\omega_0^{[m]}, \omega_\phi^{[m]}) = \int_X \log\left(\frac{\omega_\phi^{[m]}}{\omega_0^{[m]}}\right) \omega_\phi^{[m]}$$

(where $\frac{\omega_\phi^m}{\omega_0^m}$ is understood as a Radon-Nikodym derivative).

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In the **weighted case**, we need to understand the operator

$$\mathrm{MA}_v(\phi) := v(\mu_\phi)\omega_\phi^{[m]}$$

on $\mathcal{E}^1_{\mathbb{T}}$ in order to define our entropy term $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\phi))$.

**Proposition (AJL '21, Han-Li '20)**

The operator $\phi \mapsto \mathrm{MA}_v(\phi)$ extends to a Radon measure-valued operator on $\mathcal{E}^1_{\mathbb{T}}$, which is continuous along pointwise decreasing sequences in $\mathcal{E}^1_{\mathbb{T}}$ (wrt the weak topology of measures).

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**First step: the case of polynomial $v$.** We assume that $v > 0$ is of the form

$$v(\mu) = \prod_{a=1}^{k} (\langle v_a, \mu \rangle + c_a)^{n_a}.$$

We construct an associated semisimple principal fibration $(Y, \tilde{\omega}_\phi)$, which will allow us to compute $\int_X f \, \mathrm{MA}_v(\phi)$ for torus-invariant continuous functions $f$ on $X$ (as in Tran-Trung's talk).

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$$\int_Y f \, \tilde{\omega}_\phi^{[m+n]} = v_B^{-1} \int_X f \, v(\mu_\phi) \, \omega_\phi^{[m]}$$

with $v_B$ the volume of the base of the fibration.

*Recall that* $v(\mu) = \prod_{a=1}^{k} (\langle v_a, \mu \rangle + c_a)^{n_a}$*.*

Define the base of the fibration as $B = B_1 \times \cdots \times B_k$ where $B_a = (\mathbb{P}^{n_a}, \omega_a)$, with Fubini-Study metrics of scalar curvature $= 2n_a(n_a + 1)$.

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$$d\theta = \sum_{a=1}^{k} (\pi_B^* \omega_a) \otimes v_a.$$

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We then set $Y = (X \times P)/(\mathbb{T}_{X\times P})$, and define

$$\tilde{\omega}_0 := \omega_0 + \sum_{a=1}^{k}(\langle v_a, \mu_{\omega_0}\rangle + c_a)\pi_B^*\omega_a + \langle d\mu_{\omega_0} \wedge \theta\rangle.$$

It is a 2-form on $X \times P$ which descends to a $\mathbb{T}_Y$-invariant Kähler form $\tilde{\omega}_0$ on $Y$ (Tran-Trung's talk again).

**Theorem**

*We have a well-defined embedding $\iota$ of the set of $\omega_0$-integrable $\mathbb{T}$-invariant functions on $X$ into the set of $\tilde{\omega}_0$-integrable $\mathbb{T}_Y$-invariant functions on $Y$, sending smooth functions to smooth functions;*

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$$\mathcal{K}_{\mathbb{T}}(X, \omega_0) \hookrightarrow \mathcal{K}_{\mathbb{T}}(Y, \tilde{\omega}_0),\ \omega_\phi \mapsto \tilde{\omega}_{\iota(\phi)}$$

*such that the form induced by $\omega_\phi$ on $Y$ coincides with $\tilde{\omega}_{\iota(\phi)}$. (In particular, $d_1$-isometry.)*

**Theorem**

*We have the following integration formula:*

$$\int_Y f\,\tilde{\omega}_\phi^{[m+n]} = v_B^{-1}\int_X \iota(f)\,v(\mu_\phi)\,\omega_\phi^{[m]}$$

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For general $f \in C^0(X)$, we define $f^{\mathbb{T}}$ to be the $\mathbb{T}$-invariant function defined by the average of $f$ over torus orbits, and set

$$\int_X f\,\mathrm{MA}_v(\phi) := \int_X f^{\mathbb{T}}\,\mathrm{MA}_v(\phi).$$

Because $f \mapsto f^{\mathbb{T}}$ is linear, this defines a positive Radon measure by Riesz-Markov-Kakutani.

**The case of non-polynomial $v$.** We first extend the previous definition by linearity on the cone generated by positive linear combinations of polynomials of the above form. Because such polynomials are bounded on $\Delta$ one always has

$$\left| \int_X f \,\mathrm{MA}_{v_p}(\phi) - \int_X f \,\mathrm{MA}_{v_q}(\phi) \right| \leq \|v_p - v_q\|_{C^0(\Delta)} \int_X |f| \,\mathrm{MA}(\phi)$$

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$$\int_X f\,\mathrm{MA}_v(\phi) := \lim_{k\to\infty} \int_X f\,\mathrm{MA}_{v_k}(\phi).$$

This again defines a positive Radon measure, and concludes the proof of the extension of $\mathrm{MA}_v$ (+ a $C^0$-estimate).

**Lemma (continuity of $\mathrm{MA}_v$)**

If $(\phi_j)_j$ is a sequence in $\mathcal{E}^1_{\mathbb{T}}$ such that $d_1(\phi_j, \phi) \to 0$ for some $\phi \in \mathcal{E}^1_{\mathbb{T}}$, then $\mathrm{MA}_v(\phi_j) \to \mathrm{MA}_v(\phi)$ weakly.

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We again begin with the polynomial case: $v(\mu) = \prod_{a=1}^{k}(\langle v_a, \mu\rangle + c_a)^{n_a}$ and construct a fibration $Y$ as before.

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$$\int_X f\,\mathrm{MA}_v(\phi_j) = \lim_k \int_X f\,\mathrm{MA}_v(\phi_{j,k}) = v_B^{-1} \lim_k \int_Y f\,\mathrm{MA}(\phi_{j,k}) = v_B^{-1} \int_Y f\,\mathrm{MA}(\phi_j).$$

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Thus

$$\begin{aligned}\lim_j \int_X f\,\mathrm{MA}_v(\phi_j) &= v_B^{-1} \lim_j \int_Y f\,\mathrm{MA}(\phi_j) \\ &= v_B^{-1} \int_Y f\,\mathrm{MA}(\phi) \\ &= \int_X f\,\mathrm{MA}_v(\phi).\end{aligned}$$

## Lemma (continuity of $\mathrm{MA}_v$)

If $(\phi_j)_j$ is a sequence in $\mathcal{E}^1_{\mathbb{T}}$ such that $d_1(\phi_j, \phi) \to 0$ for some $\phi \in \mathcal{E}^1_{\mathbb{T}}$, then $\mathrm{MA}_v(\phi_j) \to \mathrm{MA}_v(\phi)$ weakly.

We again begin with the polynomial case: $v(\mu) = \prod_{a=1}^k (\langle v_a, \mu\rangle + c_a)^{n_a}$ and construct a fibration $Y$ as before. Note that even if the $\phi_j$ are in $\mathcal{E}^1_{\mathbb{T}}$, we can approximate with smooth $(\phi_{j,k})_k$ so that for $f \in C^0_{\mathbb{T}}(X)$ we have

$$\int_X f\,\mathrm{MA}_v(\phi_j) = \lim_k \int_X f\,\mathrm{MA}_v(\phi_{j,k}) = v_B^{-1} \lim_k \int_Y f\,\mathrm{MA}(\phi_{j,k}) = v_B^{-1} \int_Y f\,\mathrm{MA}(\phi_j).$$

Because $d_1(\phi_j, \phi) \to 0$, due to the embedding Theorem the same holds on $Y$. Therefore,

$$\mathrm{MA}^Y(\phi_j) \to_j \mathrm{MA}^Y(\phi).$$

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This holds for non-torus-invariant $f$ by considering $f^{\mathbb{T}}$ as before.

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&+ \left|\int_X f\,\mathrm{MA}_{v_i}(\phi) - \int_X f\,\mathrm{MA}_v(\phi)\right| \\
&\leq \left|\int_X f\,\mathrm{MA}_{v_i}(\phi_j) - \int_X f\,\mathrm{MA}_{v_i}(\phi)\right| + \|v_i - v\|_{C^0(\Delta)} \left(\int_X |f|(\,\mathrm{MA}(\phi_j) + \,\mathrm{MA}(\phi))\right).
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We take the limit in $j$, so that

$$\lim_j \left|\int_X f\,\mathrm{MA}_v(\phi_j) - \int_X f\,\mathrm{MA}_v(\phi)\right| \le 2\|v_i - v\|_{C^0(\Delta)} \left(\int_X |f|\,\mathrm{MA}(\phi)\right)$$

by continuity of the unweighted $\mathrm{MA}$.

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by continuity of the unweighted $\mathrm{MA}$. Then this limit is shown to be zero by taking $i \to \infty$, which implies that $\mathrm{MA}_v(\phi_j) \to_j \mathrm{MA}_v(\phi)$. This concludes the proof of the Lemma.

**Lemma (entropy approximation)**

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Because for fixed $f$, $\nu \mapsto \int_X f\, d\nu$ is continuous, the entropy is thus a supremum of a family of continuous functions, i.e. lsc. Now, by the previous Lemma, $\mathrm{MA}_v$ is $d_1$-continuous, so that the entropy is $d_1$-lsc.

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We then solve the weighted MA equation (Han-Li):

$$\mathrm{MA}_v(\phi_j) = \left( \frac{\int_X v(\mu_0)\omega_0^{[m]}}{\int_X g_j \omega_0^{[m]}} g_j \omega_0^{[m]} \right),$$

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$$\mathrm{MA}_v(\psi) = \lim_j \mathrm{MA}_v(\phi_j).$$

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Thus $u = \phi + c$ for some constant, so that up to substracting constants $u_j$ is the desired sequence.

# The $I_v$ functional and the weighted $d_1$-distance.

We define the weighted length of a smooth curve $t \mapsto \phi_t \in \mathcal{K}_{\mathbb{T}}$: for $v \in C^\infty(\Delta)_{>0}$,

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among all smooth curves joining $\phi_0$ and $\phi_1$ in $\mathcal{K}_{\mathbb{T}}$. It is a distance, and in fact equivalent to the $d_1$-distance:

**Lemma**

*If $v > 0$ there exists $C > 0$ such that for all $\phi_0$, $\phi_1 \in \mathcal{K}_{\mathbb{T}}$:*

$$C^{-1} d_1(\phi_0, \phi_1) \le d_{1,v}(\phi_0, \phi_1) \le C \, d_1(\phi_0, \phi_1).$$

(This is due to the fact that $v$ is bounded on $\Delta$.)

It is closely related to the $I_v$ functional defined by its variation:

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It follows that $I_v$ is $d_{1,v}$-Lipschitz: given a curve $\phi_t$ joining $\phi_0$, $\phi_1$ in $\mathcal{K}_{\mathbb{T}}$, one has

$$\begin{aligned} |I_v(\phi_0) - I_v(\phi_1)| &= \left| \int_0^1 (d_{\phi_t} I_v)(\dot{\phi}_t) \right| \\ &\leq \int_0^1 \left( \int_X \left| \dot{\phi}_t \right| v(\mu_{\phi_t}) \, \mathrm{MA}(\phi_t) \right) \leq \ell_{1,v}(\{\phi_t\}_t), \end{aligned}$$

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By the previous Lemma, this is smaller than a constant times $d_1(\phi_0, \phi_1)$, hence $I_v$ is $d_1$-Lipschitz. This allows us to extend it to $\mathcal{E}^1_\mathbb{T}$. Note that $I_v$ is by definition linear in $v$, which furthermore allows us to extend it by linearity to nonpositive weights.

# A few words on $I_v^\rho$.

Let $\rho$ be an invariant closed $(1,1)$-form. We again define $I_v^\rho$ by its variation

$$(d_\phi I_v)(\dot\phi) := \int_X \dot\phi(v(\mu_\phi)\rho \wedge \omega_\phi^{[m-1]} + \langle (dv)(\mu_\phi), \mu_\rho\rangle \omega_\phi^{[m]}).$$

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One has the following:

**Proposition**

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The proof relies on somewhat tedious computation, based on [BDL]. The key is to obtain an explicit expression for $I_v^\rho(\phi_1) - I_v^\rho(\phi_0)$ (which brings us to the next page...)

$$
\begin{aligned}
& I_v^\rho(\phi_1) - I_v^\rho(\phi_0) \\
& = \int_X (\phi_1 - \phi_0) \left( \sum_{j=0}^{m-1} \left[ \int_0^1 s^j (1-s)^{m-1-j} v(s\mu_1 + (1-s)\mu_0) ds \right] \rho \wedge \omega_1^{[j]} \wedge \omega_0^{[m-j-1]} \right) \\
& + \int_X (\phi_1 - \phi_0) \left( \sum_{j=0}^{m-1} \left\langle \int_0^1 s^j (1-s)^{m-1-j} (dv)(s\mu_1 + (1-s)\mu_0) ds, \mu_\rho \right\rangle \omega_1^{[j]} \wedge \omega_0^{[m-j]} \right).
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&= \int_X (\phi_1 - \phi_0) \left( \sum_{j=0}^{m-1} \left[ \int_0^1 s^j (1-s)^{m-1-j} v(s\mu_1 + (1-s)\mu_0) ds \right] \rho \wedge \omega_1^{[j]} \wedge \omega_0^{[m-j-1]} \right) \\
&+ \int_X (\phi_1 - \phi_0) \left( \sum_{j=0}^{m-1} \left\langle \int_0^1 s^j (1-s)^{m-1-j} (dv)(s\mu_1 + (1-s)\mu_0) ds, \mu_\rho \right\rangle \omega_1^{[j]} \wedge \omega_0^{[m-j]} \right).
\end{aligned}
$$

For example, for the $C^1$-estimate, one uses linearity: $I_v^\rho(\phi) - I_w^\rho(\phi) = I_{v-w}^\rho(\phi)$. Then, this formula for $\phi_1 = \phi$, $\phi_0 = 0$ allows us to have an estimate of the form

$$
\begin{aligned}
|I_{v-w}^\rho(\phi)| &\leq C\|v-w\|_{C^1(\Delta)} \int_X \sum |\phi| \omega_\phi^{[j]} \wedge \omega_0^{[m-j]} \\
&\leq C'\|v-w\|_{C^1(\Delta)} \int_X |\phi| \omega_\phi^{[m]},
\end{aligned}
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as desired.

# Conclusion

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• We have to show that it is the largest such extension. By the entropy approximation Lemma, given $\phi \in \mathcal{E}^1_{\mathbb{T}}$, one can find a sequence $(\phi_j)_j$ in $\mathcal{K}_{\mathbb{T}}$ conveging in $d_1$ and in weighted entropy to $\phi$. On the other hand, $I_w$ and $I^\rho_v$ are $d_1$-continuous, so that

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• Regarding linearity in $v$, $w$: since all the other components are linear, we need only look at the entropy. In fact we will need the additional "constant" term:

$$Ent(\omega_0^{[m]}, \mathrm{MA}_v(\phi)) - \int_X \log(v(\mu_0))v(\mu_0)\omega_0^{[m]} = \int_X \log\left(\mathrm{MA}(\phi)/\omega_0^m\right) \mathrm{MA}_v(\phi),$$

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• Regarding geodesics: Lahdili proved that $\mathrm{M}_{v,w}$ is convex along Mabuchi geodesics in $\mathcal{K}_{\mathbb{T}}$ (closely following Berman-Berndtsson). The general result then follows from entropy approximation.

Fin.