Goal of the reading seminar:

[AJL]: Apostolov, Jubert, Lahdili,
Weighted K-stability and coercivity with applications to extremal Kähler and Sasaki metrics.

Weighted cscK metrics / semisimple principal fibration

Motivating questions: on a given Kähler manifold,

- When does there exist extremal Kähler metrics in the sense of Calabi? (not always)
- Are extremal Kähler metrics well behaved wrt the geometry of the manifold? (expected)

Here fiber bundles

Definition: $\begin{matrix} Y \\ \downarrow \pi \\ B \end{matrix}$ is a fiber bundle with fiber X if

$\forall x \in B$, $\exists$ U open neighborhood of x in B st

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, phantom, "\simeq"] \arrow[dr, "\pi"'] & U \times X \arrow[d, "pr_1"] \\ & U \end{tikzcd}$$

in the regularity desired: topological, differentiable, holomorphic

Examples: line bundles, vectorbundles

Baby example: $Bl_0\mathbb{P}^2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(1)) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(-1))$

$\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

also: projective compactification of $\mathcal{O}(-1)$ (adding a $\mathbb{P}^1$ at ∞)

$$\begin{tikzcd}
\mathcal{O}(-1) = Bl_0\mathbb{C}^2 = \{(z,w,[s:t]) \in \mathbb{C}^2\times\mathbb{P}^1 \mid zt = ws\} \arrow[r, "\text{proj}_1", "\text{blowup}"'] \arrow[d, "\text{proj}_2"', "\text{bundle map}"] & \mathbb{C}^2 \\
\mathbb{P}^1 &
\end{tikzcd}$$

$Bl_0\mathbb{C}^2$ obtained by gluing $\mathbb{C}^2_{(t,x)}$ to $\mathbb{C}^2_{(s,y)}$ along $\mathbb{C}^*\times\mathbb{C}$ by

$$(t,x)\sim\left(\frac{1}{t}, tx\right)$$

$Bl_0\mathbb{P}^2$ ———— $\underset{(t,[x:y])}{\mathbb{C}\times\mathbb{P}^1}$ to $\mathbb{C}\times\mathbb{P}^1$ by $(t,[x:y])\sim\left(\frac{1}{t},[tx:y]\right)$

$$\underbrace{\mathbb{C}^2\setminus\{0\}}_{\text{open subset}} \subset Bl_0\mathbb{C}^2 \subset Bl_0\mathbb{P}^2$$

Key insight, originating from Calabi, developed a lot by Apostolov & collaborators

existence of extremal Kähler metric on the fiber bundle Y

$\Updownarrow$

existence of certain weighted cscK metrics on the fiber X

Want to show, using $\underbrace{\text{[Calabi 82]}}$'s methods, how this relationship appears.

↳ $Bl_0\mathbb{P}^2$, Hirzebruch surfaces
& higher dim analogous

Y^n compact Kähler manifold, α Kähler class on Y

Definition [Calabi 82]: $\omega \in \alpha$ is extremal if it minimizes the L^2 norm of the scalar curvature among Kähler metrics in α.

Recall: • ω Kähler $\iff$ 2-form that locally writes as

$$\omega \underset{loc}{=} i\partial\bar{\partial}\varphi = i\sum_{j,k} \underbrace{\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}}_{\text{positive definite Hermitian}} dz_j \wedge d\bar{z}_k$$

- Ricci form $\quad Ric(\omega) \underset{loc}{=} i\partial\bar{\partial}\left(-\ln\det\left(\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k}\right)\right)$ global closed real (1,1) form

- Scalar curvature $\quad S(\omega) = n\,\frac{Ric(\omega)\wedge\omega^{n-1}}{\omega^n}$ function $Y \to \mathbb{R}$

- L^2 norm of scalar curv : $\quad \int_Y S(\omega)^2\,\omega^n$

Theorem [Calabi 82]: $\omega \in \alpha$ is extremal iff $\operatorname{grad} S(\omega)$ is a holomorphic vector field.

Corollary: A cscK metric ($S(\omega)$ constant) is extremal.

Calabi showed in that same paper that $Bl_0\mathbb{P}^2$ admits extremal, non cscK metrics.

Why no cscK?

Theorem [Matsushima Lichnerowicz]: If ω is a cscK metric on Y, $\mathrm{Aut}(Y)$ group of biholom of Y, then $\mathrm{Isom}(\omega)$ is a maximal compact subgp in $\mathrm{Aut}(Y)$, and more precisely $\mathrm{Aut}(Y) = \mathrm{Isom}(\omega)^{\mathbb{C}}$.

Example:
$$\mathrm{Aut}^0(Bl_0\mathbb{P}^2) = \mathrm{Stab}_{\mathrm{Aut}(\mathbb{P}^2)}(\{0\}) = \underset{\underset{U(2)^{\mathbb{C}} = GL_2(\mathbb{C})}{\cup}}{\left\{ \overline{\begin{bmatrix} * & \circledast & \circledast \\ 0 & * & * \\ 0 & * & * \end{bmatrix}} \right\}} \subset \overset{\overset{\mathrm{Aut}(\mathbb{P}^2)}{\shortparallel}}{PGL_3(\mathbb{C})}$$

maximal compact subgroup is $\underset{\underset{U(2)}{\cong}}{\left\{ \overline{\begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}} \right\}} \subset PU(3)$

$$2\dim_{\mathbb{R}} U(2) = \dim_{\mathbb{C}} GL_2(\mathbb{C}) < \dim_{\mathbb{C}} \mathrm{Aut}^0(Bl_0\mathbb{P}^2)$$

In 82, there was no Matsushima thm for extremal Kähler metrics. But Calabi worked as if it were the case.

→ Calabi looks for extremal metrics among $U(2)$-invariant metrics

$\longrightarrow$ consider the restriction to $\mathbb{C}^2 \setminus \{0\} \subset Bl_0\mathbb{P}^2$ where action of $U(2)$ is the standard linear one.

$\longrightarrow$ consider $\omega = i\partial\bar\partial\varphi$ where $\varphi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{R}$ psh $U(2)$-inv

$U(2)$-inv $\rightsquigarrow$ $\varphi(z_1, z_2) = u(\ln\|z\|^2)$

$\longrightarrow$ compute the scalar curvature

$\omega = i\partial\bar\partial\varphi$

$$\frac{\partial}{\partial\bar z_1}\left(u\left(\ln(z_1\bar z_1 + z_2\bar z_2)\right)\right) = \frac{z_1}{\|z\|^2} u'(\ln\|z\|^2)$$

$$\frac{\partial^2}{\partial z_1 \partial\bar z_1}\left(u\left(\ln(z_1\bar z_1 + z_2\bar z_2)\right)\right) = \frac{1}{\|z\|^2} u'(\ln\|z\|^2) - \frac{|z_1|^2}{\|z\|^4} u'(\ln\|z\|^2) + \frac{|z_1|^2}{\|z\|^4} u''(\ln\|z\|^2)$$

...

write $t = \ln\|z\|^2$

$$\omega = e^{-t}u'(t)\left(idz_1 \wedge d\bar z_1 + idz_2 \wedge d\bar z_2\right) + e^{-2t}\left(u''(t) - u'(t)\right)\left(\sum_{j,k} z_j \bar z_k \, idz_j \wedge d\bar z_k\right)$$

$$\omega^2 = e^{-2t}u'(t)u''(t)\left(idz_1 \wedge d\bar z_1 \wedge idz_2 \wedge d\bar z_2\right)$$

$$Ric(\omega) = i\partial\bar\partial\left(-\ln\det\frac{\partial^2\varphi}{\partial z_j \partial\bar z_k}\right) = i\partial\bar\partial\, v(t)$$

$$v(t) = -\ln\left(e^{-2t}u'(t)u''(t)\right)$$

$$S(\omega) = 2\,\frac{Ric(\omega)\wedge\omega}{\omega^2} = \frac{v'(t)}{u'(t)} + \frac{v''(t)}{u''(t)}$$

cscK equation reduce to a 1-variable ODE

(similarly for the extremal equation up to pushing the computations further)

still need to check that solution u comes from a global Kähler metric

How does this relate to the fiber $\mathbb{P}^1$?

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\text{fiber}"] & \mathbb{C}^2\setminus\{0\} \arrow[r] \arrow[d, phantom, "\cap"] & \mathbb{P}^1 \\
\mathbb{P}^1 \arrow[r, phantom, "\subset"] & Bl_0\mathbb{P}^2 \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

in particular fiber over $[1:0]$ in $\mathbb{C}^2\setminus\{0\}$ is $\mathbb{C}^*\times\{0\}$

Restriction of φ to $\mathbb{C}^*\times\{0\}$ provides a Kähler metric on $\underset{\underset{\mathbb{P}^1\text{ fiber}}{\cap}}{\mathbb{C}^*\times\{0\}}$

(actually, the restriction fully determines φ).

$$\omega_X = \frac{\partial^2}{\partial z\partial\bar z}\left(\varphi(z,0)\right) i\,dz\wedge d\bar z \quad \text{on } \mathbb{C}^*$$

$$= \frac{1}{z\bar z}\, u''(\ln z\bar z)\, i\,dz\wedge d\bar z$$

$$Ric(\omega_X) = \frac{1}{z\bar z}\left(\ln u''\right)''(\ln z\bar z)\, i\,dz\wedge d\bar z$$

$$S(\omega_X) = \frac{Ric(\omega_X)}{\omega_X} = \frac{(\ln u'')''}{u''}$$

$S(\omega_Y) = \frac{v'}{u'} + \frac{v''}{u''}$ where $v(t) = 2t - \ln u'(t) - \ln u''(t)$

highest order term in $S(\omega_Y)$ is $S(\omega_X)$

$\frac{v'}{u'} + \frac{v''}{u''} = cst$ baby example of a weighted cscK eqn on $\mathbb{P}^1$. [AJL]

This type of equations is very general and encompasses:

→ will be defined next week

- usual cscK, extremal, KE metrics
- g-solitons (Kähler-Ricci solitons, Mabuchi solitons, ..)

  $\cup$

  Sasaki Einstein metrics

Main theorems in [AJL]:

Theorem 1 [AJL]: ω_X weighted cscK metric on X

$$\Downarrow$$

the corresponding weighted Mabuchi functional is coercive.

→ $T^{\mathbb{C}}$-coercive on the space of T-invariant Kähler metrics in $[\omega_X]$

why a torus? e.g. for extremal, recall grad $S(\omega)$ is a holom v.f.
This v.f. generates a torus T. associated Weighted cscK eqn makes sense for T-inv K metrics.

Idea of proof: follow and adopt

Darvas Rubinstein + Berman Darvas Lu

for standard cscK metrics.

requires:
- extending weighted Mabuchi functional to finite energy metrics
- proving regularity of minimizers.

Theorem 2 [AJL]: ω_X weighted cscK $\Longrightarrow$ $(X,[\omega_X])$ is uniformly weighted K-stable

Idea: use thm 1 + consider slopes of the weighted Mabuchi along certain geodesic rays. [Boucksom Hisamoto Jonsson] [Sjöström Dyrefelt]

Theorem 3 [AJL]: Let (Y,ω_Y) be a semisimple principal fibration with fiber (X,ω_X). Then the following are equivalent:

① $(Y,[\omega_Y])$ extremal

② $(X,[\omega_X])$ weighted cscK (for weights coming from the construction)

③ weighted Mabuchi functional on X is coercive.

③ ⇒ ② converse to thm 1 in this specific situation

① ⇒ ② shows that extremal metric reflect the fibration structure.

Proof of ③ ⇒ ① uses cleverly Chen-Cheng's continuity method.
+ He

<u>Thm 4</u> [Jubert]: (Y, ω_Y) semisimple principal fibration with toric fiber (X, ω_X). Then $\exists$ extremal on Y is equivalent to the moment polytope Δ of (X, ω_X) being weighted uniformly K-stable.

⇝ a version of the uniform YTD conjecture.

condition looks like: $2\int_{\partial\Delta} f \, v \, d\sigma - \int_{\Delta} f \, w \, dx \geq \lambda \|f\|$ for convex functions $f : \Delta \to \mathbb{R}$

v, w weights.