# Coercivity Principle.

## I) Notation

- $\mathbb{T} \subset Aut_r(X)$ - fixed connected compact torus
- $G = \mathbb{T}^{\mathbb{C}} \subset Aut_r(X)$ corresponding complex torus

Then, we have

- $L_1$ - length on $\mathcal{K}(X, \omega_0)$ :

$$L_1(\varphi_t) = \int_0^1 \left( \int_X |\dot{\varphi}_t| \omega_{\varphi_t}^{[m]} \right) dt$$

$(\varphi_t)_t$ : smooth curve in $\mathcal{K}(X, \omega_0)$

- $d_1$ - distance on $\mathcal{K}(X, \omega_0)$

$$d_1(\phi_0, \phi_1) = \inf_t \{ L_1(\psi_t) \mid \psi_t \in \mathcal{K}(X, \omega_0) \\ \psi_0 = \phi_0 \,;\, \psi_1 = \phi_1 \}$$

- $d_1$ – distance on $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ defined by taking smooth curve on $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$

<u>Fact</u>

$(\mathcal{K}(X, \omega_0), d_1)$ is a metric space & $(\mathcal{K}_{\mathbb{T}}(X, \omega_0), d_1)$ is a metric subspace of $(\mathcal{K}(X, \omega_0), d_1)$

- $\mathbb{I}$ & $\mathbb{J}$-functionals on $\mathcal{K}(X, \omega_0)$

$$(d_\varphi \mathbb{I})(\dot{\varphi}) = \int_X \dot{\varphi}\, \omega_\varphi^{[m]} \,;\, \mathbb{I}(0) = 0 .$$

$$\mathbb{J}(\varphi) = \int_X \varphi\, \omega_0^{[m]} - \mathbb{I}(\varphi)$$

<u>Rmk</u>: • $\mathbb{I}(\varphi + c) = \mathbb{I}(\varphi) + c\,\mathrm{Vol}(X, \omega_0)$

• $\mathbb{J}(\varphi + c) = \mathbb{J}(\varphi) =: \mathbb{J}(\omega_\varphi)$

• $\mathbb{J}(\omega_\varphi) \geq 0$ ; " $=$ " $\Leftrightarrow \omega_\varphi = \omega_0$

• $\mathring{\mathcal{K}}(X, \omega_0) = \mathcal{K}(X, \omega_0) \cap \mathbb{I}^{-1}(\{0\})$

• $G = \mathbb{T}^{\mathbb{C}} \curvearrowright \mathcal{K}(X, \omega_0)$ preserving $\mathbb{T}$- invariant Kähler metrics.

$\to G \curvearrowright \mathring{\mathcal{K}}(X, \omega_0)$ & $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$

st $\boxed{\omega_{\sigma \cdot \varphi} = \sigma^* (\omega_\varphi)}$ $\forall\, \sigma \in G$

* $\varphi \in \mathring{\mathcal{K}}(X, \omega_0)$

$\to$ $d_1^G$ - rel. distance on $\mathring{\mathcal{K}}$, $\mathring{\mathcal{K}}_\pi$.

$$d_1^G(\varphi_0, \varphi_1) = \inf_{b_0, b_1 \in G} d_1(b_0 \cdot \varphi_0, b_1 \cdot \varphi_1)$$

(Darvas - Rubinstein) $d_1^G$ is $G$-invariant.

$$\Rightarrow \quad d_1^G(\varphi_0, \varphi_1) = \inf d_1(\varphi_0, b \cdot \varphi_1).$$

(*) $\boxed{\exists\, b \in G = d_1(\varphi_0, b \cdot \varphi_1)}$

<u>Definition:</u> $F$ a functional on $\mathcal{K}_\pi(X, \omega_0)$, $F$ is <u>$G$-coercive</u> if

$\exists$ uniform $(\lambda, \delta)$ s.t

$$\mathcal{F}(\varphi) \geq \lambda d_1^G(0, \varphi) - \delta$$

$$\forall \varphi \in \mathring{K}_{\mathbb{T}}.$$

Prop: (Darvas - Rubinstein)

$\mathcal{F}$ is $G$-coercive $\iff$ $\exists$ uniform $(\lambda, \delta)$ s.t

$$\mathcal{F}(\varphi) \geq \lambda \inf_{g \in G} \mathbb{J}(g^* \omega_\varphi) - \delta$$

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(Darvas)

$$\boxed{(K, d_1) = (\mathcal{E}^1(X, \omega_0), d_1)}$$

$$\mathcal{E}^1(X,\omega_0) = \left\{ \varphi \in PSH(X,\omega_0) \;\middle|\; \int_X MA(\varphi) = \int_X \omega_0^{[n]} \right.$$

$$\left. \& \int_X |\varphi|\, MA(\varphi) < \infty \right\}$$

<u>Fact</u>

- $(\mathcal{E}^1_\Pi, d_1)$ is a $\boxed{\text{geodesic space}}$.

ie, $\forall\, \varphi_0, \varphi_1 \in \mathcal{E}^1_\Pi$, $\exists \{\varphi_t\} \subset \mathcal{E}^1_\Pi$ connecting $\varphi_0$ & $\varphi_1$, loc. distance minimiser

- $\{\varphi_t\}$ is called <u>weak geodesic</u>.

= limit of $C^{1,\bar{1}}$-ged btw. element in $K_\Pi$.

$\hookrightarrow$ $\varphi_t \in \mathcal{E}^1_\Pi \cap C^{1,\bar{1}}([0,1] \times X)$

## III Coercivity principle

<u>Setting</u>: $\mathcal{F} : \mathcal{K}_{\mathbb{T}}(X, \omega_0) \longrightarrow \mathbb{R}$

l.s.c wrt $d_1$ on $\mathcal{K}_{\mathbb{T}}$

- $\mathcal{F} : \mathcal{E}^1_{\mathbb{T}} \longrightarrow \mathbb{R} \cup \{+\infty\}$ largest lower semicontinuous extension:

$$\mathcal{F}(\phi) = \sup_{\varepsilon > 0} \left( \inf_{\substack{\psi \in \mathcal{K}_{\mathbb{T}} \\ d_1(\psi, \phi) < \varepsilon}} \mathcal{F}(\psi) \right)$$

<u>Thm</u> (Coercitivity principle, Darvas–Rubinstein)

Suppose that $\begin{cases} \mathcal{F}(\varphi + c) = \mathcal{F}(\varphi) \\ \mathcal{F}(g.\varphi) = \mathcal{F}(\varphi) \end{cases}$

$\forall\, g \in G$.

And $\mathcal{F}$ satisfie the following conditions:

(1) (Convexity) For $\varphi_0, \varphi_1 \in \mathcal{K}_\pi$ & $(\varphi_t)_{t\in[0,1]}$ $d_1$-geodesic joining $\varphi_0$ & $\varphi_1$, then

$t \to \mathcal{F}(\varphi_t)$ is $\left\{ \begin{array}{l} \text{continuous} \\ \text{convex.} \end{array} \right.$

(2) (Uniqueness) $G$ acts transitively on $\mathcal{M}$ = Set minimizers of $\mathcal{F}$.

(3) (Compactness) If $\{\varphi_j\}_j \subset \mathcal{E}^1_\pi$ satisfies

$$\lim_{j\to+\infty} \mathcal{F}(\varphi_j) = \inf_{\mathcal{E}^1_\pi} \mathcal{F}$$

and $d_1(0, \varphi_j) \leq C$, for some $C > 0$

then $\exists\ \varphi \in \mathcal{E}^1_\pi$ & $\{\varphi_{j_k}\}_k$ subseq

such that $\varphi_{j_k} \xrightarrow{d_1} \varphi$.

Then TFAE

(i) $\mathcal{M}$ - (set of minimisers of $\mathcal{F}$ in $\mathcal{E}^1_\pi$) is not empty

(ii) $\mathcal{F}$ is $G$ - coercive.

In particular: If in addition

(4) $\forall$ minimiser of $\mathcal{F}$ is smooth

then

$\mathcal{F}$ is $G$-coercive $\Leftrightarrow$ $\mathcal{F}$ has minimizer in $K_T$
(Regularity)

Rmk: • Berman-Darvas-Lu proved:
Assume that $\exists$ cscK metric
then mininiser of Mabuchi fct
is smooth

• Chen-Cheng'21 proved that
Mabuchi is coercive $\Longrightarrow$ mininuiser
of Mabuchi energy is smooth.

In the AJL paper:
If $\exists$ $(v, w_0)$-ext metric

then $\mathcal{M}_{v,w}$ is $G$ - coercive

with $w = \ell^{ext}_{v, w_0} w_0$

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TODAY: prove the coercive principle of Darvas - Rubinstein

Next talks by Chinh & Rémi

- extension of $\mathcal{M}_{v,w}$ on $\mathcal{E}^1_\pi$
- prove compactness ($\sim$ BDL)
- prove convexity (Lahdili)

* Regularity ($\sim$ BDL)

IV Proof of Coercivity principle

(ii) $\Rightarrow$ (i) $G$-coercivity implies

$\mathcal{M} \neq \phi.$

- Since $\mathcal{F}$ is $G$-coercive :

$$\mathcal{F}(\varphi) \geq \lambda\, d_1^G(0,\phi) - \delta$$

$$\forall \phi \in \mathcal{K}_{\Pi}$$

$\Rightarrow$ $\mathcal{F}$ is bdd from below

$\Rightarrow$ $\exists\, \phi_j \in \mathcal{K}_{\Pi}$ s.t

and

$$d_1(0,\phi_j) \leq d_1^G(0,\phi_j) + \varepsilon$$

$$\leq C$$

"Compactness" $\exists\, (\phi_{j_k})$ , $\phi_{j_k} \to \phi$

in $\mathcal{E}^1_{\Pi}$ $\Rightarrow$ $\phi$ is minimiser of $\mathcal{F}$. ( so if we have "Regularity", $\Rightarrow \phi \in K_{\Pi}$) $\square$

(i) $\Rightarrow$ (ii) $\mathcal{M} \neq \phi$

$\Rightarrow$ $\mathcal{F}$ is $G$-coercive

Since $\mathcal{M} \neq \phi$, taking $\boxed{\phi \in \mathcal{M} \text{ is minimiser}}$ of $\mathcal{F}$.

- Denote

+ $\mathcal{F}(\phi, \psi) = \mathcal{F}(\psi) - \mathcal{F}(\phi) \geq 0$

+ $C := \inf \left\{ \dfrac{\mathcal{F}(\phi, \psi)}{d_1^G(\phi, \psi)} \,\middle|\, \psi \in K_{\Pi} \right.$

$\& \, d_1^{G}(\psi, \phi) \geq 1 \}$

- If $C > 0 \Rightarrow$ we are done.
- Suppose $C = 0$, $\Rightarrow \exists \, (\psi_k) \subset K_{\pi}$

s.t.

$$\frac{F(\phi, \psi_k)}{d^{G}(\phi, \psi_k)} \xrightarrow[k \to +\infty]{} 0$$

- Taking $(\psi_k(t)) \subset \mathcal{E}^1_{\pi}$

unit speed $-d_1-$geodesic

s.t.

$$\begin{cases} \psi_k(0) = \phi \\ \psi_k(d_1(\phi, \psi_k)) = \psi_k \end{cases}$$

and

$$t \longmapsto F(\psi_k(t)) \text{ is convex}$$

(by "convexity"-condition):

- By convexity:

$$0 \leq \frac{\mathcal{F}(\psi_k(1)) - \mathcal{F}(\phi)}{1} \leq \frac{\mathcal{F}(\phi) - \mathcal{F}(\psi_k)}{d_1(\phi, \psi_k)} = \frac{\mathcal{F}(\phi, \psi_k)}{d_1(\phi, \psi_k)}.$$

$$\xrightarrow[k \to +\infty]{} 0$$

$\Rightarrow$ by passing a subseq of $(\psi_k(1))$, we have

$$d_1(\psi_k(1), \tilde{\phi}) \to 0 \quad (k \to +\infty)$$

for some $\tilde{\phi} \in \mathcal{H}$ (minimiser)

Since $G$ acts on $\mathcal{H}$ transitively

$\Rightarrow \exists\ h \in G$ st

$$\tilde{\phi} = h.\phi$$

$$\Rightarrow \quad 0 = d_1(h.\phi, \tilde{\phi})$$

$$\geq \boxed{d_1(h.\phi, \psi_k(1))} - d_1(\psi_k(1), \tilde{\phi}) \qquad (*)$$

Moreover

$$d_1^G(\phi, \psi_k(1)) = d_1(\phi, \psi_k(1))$$
$$= 1$$

because $(\psi_k(t))$ is $d_1$-geodesic

$$\Rightarrow \boxed{d_1(h.\phi, \psi_k(1)) \geq 1}.$$

since $\quad d_1(\psi_k(1), \tilde{\phi}) \to 0$

$(*) \quad \Rightarrow \quad 0 = d_1(b.\phi, \tilde{\phi}) \geq 1$

$\lightning \quad \square$

## II - Weighted Functionals . $X$ - kah.

- Let $\omega_0$ is $\mathbb{T}$- inv. Kähler metric
- $\mu : X^m \longrightarrow \Delta \subset \mathfrak{t}^*$

  momentum map

  $v : \Delta \longrightarrow \mathbb{R}^{>0}$

  $w : \Delta \longrightarrow \mathbb{R}$

- Weighted Mabuchi energy.

$$M_{v,w}(\varphi) = \int_X \log\left(\frac{v \circ \mu_\varphi \, \omega_\varphi^m}{\omega_0^m}\right) v_{\circ}\mu_\varphi \, \omega_\varphi^{[m]}$$

$$- 2\,\mathbb{I}_v^{Ric(\omega_0)} + \mathbb{I}_w(\varphi) - \int_X \log(v \circ \mu_0) v \circ \mu_0 \, \omega_0^{[m]}$$

Definition of $\mathbb{I}_\vartheta^\theta$, $\mathbb{I}_W$, $\mathbb{J}_\vartheta$.

(Aubin - Mabuchi Functionals)

- $\mathbb{I}_\vartheta : K_{\mathbb{T}} \longrightarrow \mathbb{R}$

$$(d_\varphi \mathbb{I})(\dot{\varphi}) = \int_X \dot{\varphi}\, \vartheta \circ \mu_\varphi\, \omega_\varphi^{[m]}$$

$$\mathbb{I}_\vartheta(0) = 0$$

- $$\mathbb{J}_\vartheta = \int_X \varphi\, \vartheta \circ \mu_0\, \omega_0^{[m]} - \mathbb{I}_\vartheta(\varphi)$$

- $\theta$ - a fixed $\mathbb{T}$-inv closed $(1,1)$ form

assume $\exists$ momentum map $\mu_\theta : X \to \mathfrak{t}^*$

$$\mathbb{I}_\vartheta^\theta : K_{\mathbb{T}} \longrightarrow \mathbb{R}$$

$$(d_\varphi I_\vartheta^\theta)(\dot\varphi) = \int \dot\varphi \Big[ \vartheta_0 \mu_\varphi\, \theta \wedge \omega_\varphi^{[n-1]}$$

$$+ \langle (d\vartheta)(\mu_\varphi), \mu_\theta \rangle\, \omega_\varphi^{[n]} \Big]$$

$\vartheta = 1$ and $w = \frac{n[\theta][\omega]^{n-1}}{[\omega]^n}$

$$\Rightarrow \underbrace{-2 I_\vartheta^\theta + I_w}_{\mathcal{J}_w^\theta \text{ - Chen } X} = \int \dot\varphi \Big[ \theta \wedge \omega_\varphi^{(n-1)} - \frac{n[\theta].[\omega]^{n-1}}{[\omega]^n} \omega_\varphi^{n} \Big].$$

- $\vartheta \equiv 1 \Rightarrow \begin{cases} I_1 = I & \text{Aubin - fct} \\ J_1 = J \end{cases}$

- <u>Lemma</u> $\exists\ C := C(X, \omega_0, \vartheta) > 0$
- $\frac{1}{C} J(\varphi) \leqslant J_\vartheta(\varphi) \leqslant C J(\varphi)$
- $|J_\vartheta(\varphi) - J_w(\varphi)|$

$$\leq \|v-w\|_{C^0(\Delta)} \bar{J}_1(\varphi)$$

$$\bullet \ \|I_v(\varphi) - I_w(\varphi)\| \leq \|v-w\|_{C^0} \left( \|\varphi\|_{L^1} + \bar{I}_1(\varphi) \right)$$

$$\begin{cases} a_1 \sim \bar{J}_v \\ \quad \sim \bar{J} \end{cases}$$

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