Weighted cscK metrics

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Delcroix's Reading group

Constant scalar curvature Kähler metrics

Normalize ω by $\int_X \omega^n = 1$.

- The scalar curvature of ω is a smooth real function

$$S(\omega) := n\frac{Ric(\omega) \wedge \omega^{n-1}}{\omega^n}.$$

- ω is a cscK metric if

$$S(\omega) = \bar{S} = n\int_X c_1(X) \wedge [\omega]^{n-1}.$$

- Mabuchi: $\exists$ a functional on $\mathcal{H}$ having cscK metrics as crit. points

$$\frac{d}{dt}\Big|_{t=0} \mathcal{M}(u + t\psi) = \frac{1}{vol(\omega)}\int_X \psi(\bar{S} - S(\omega_u)\omega_u^n.$$

Mabuchi K-energy

The explicit formula was found by Chen, Tian

$$\mathcal{M}(u) = \bar{S}\mathrm{I}(u) - n\mathrm{I}^{Ric(\omega)}(u) + Ent(\omega^n, \omega_u^n),$$

$$\mathrm{I}(u) = \frac{1}{(n+1)} \sum_{k=0}^{n} \int_X u\, \omega_u^k \wedge \omega^{n-k},$$

$$Ent(\omega^n, \omega_u^n) = \int_X \log\left(\frac{\omega_u^n}{\omega^n}\right) \omega_u^n.$$

For a real $(1,1)$-form ρ the contracted energy I^ρ is defined by

$$\mathrm{I}^\rho(u) = \frac{1}{n} \sum_{k=0}^{n-1} \int_X u\, \rho \wedge \omega_u^k \wedge \omega^{n-k-1}.$$

Regularity of minimizers of $\mathcal{M}$

Goal of today is to prove

Theorem 1

Assume ω is cscK, u is a $\mathcal{E}^1$ minimizer of $\mathcal{M}$. Then u is smooth.

Monge-Ampère energy

We have

$$\mathrm{I}(u) - \mathrm{I}(v) = \frac{1}{n+1} \sum_{k=0}^{n} \int_X (u-v)\, \omega_u^k \wedge \omega_v^{n-k},$$

$$\int_X (u-v)\, \omega_u^n \leq \int_X (u-v)\, \omega_u^k \wedge \omega_v^{n-k} \leq \int_X (u-v)\, \omega_v^n,$$

$$\int_X (u-v)\omega_u^n \leq \mathrm{I}(u) - \mathrm{I}(v) \leq \int_X (u-v)\omega_v^n.$$

If $\rho \geq 0$ and ρ is closed, then

$$\int_X (u-v)\rho \wedge \omega_u^{n-1} \leq \mathrm{I}^\rho(u) - \mathrm{I}^\rho(v) \leq \int_X (u-v)\, \rho \wedge \omega_v^{n-1}.$$

The Aubin $\mathcal{I}$ functional

Always normalize potentials by $\mathrm{I} = 0$. The Aubin $\mathcal{I}$ functional is defined by

$$\mathcal{I}(u, v) := \int_X (u - v)(\omega_v^n - \omega_u^n) \geq 0.$$

It is non degenerate: $\mathcal{I}(u, v) = 0$ implies $u = v$.
Strong topology: $\mathcal{I}(u_j, u) \to 0$ is equivalent to $d_1(u_j, u) \to 0$.
It satisfies a quasi-triangle inequality (BBEGZ):

$$\mathcal{I}(u, v) \leq C(\mathcal{I}(u, w) + \mathcal{I}(w, v)).$$

The Aubin $\mathcal{I}$ functional

Lemma 1

$$\frac{1}{n(n+1)}\mathcal{I}(u,v) \leq \mathrm{I}^{\omega_u}(v) - \mathrm{I}^{\omega_u}(u) \leq \frac{1}{n}\mathcal{I}(u,v).$$

Proof.

Recalling $\mathrm{I}(u) = \mathrm{I}(v) = 0$, we have

$$\mathrm{I}^{\omega_u}(v) - \mathrm{I}^{\omega_u}(u) = \frac{1}{n}\sum_{k=0}^{n-1}\int_X (v-u)\omega_v^k \wedge \omega_u^{n-k-1} \wedge \omega_u$$

$$= -\frac{1}{n}\int_X (v-u)\omega_v^n.$$

□

The Aubin $\mathcal{I}$ functional

Also have

$$\int_X (v-u)\omega_v^n \leq 0 \leq \int_X (v-u)\omega_u^n.$$

It follows that

$$-\int_X (v-u)\omega_v^n \leq \int_X (v-u)(\omega_u^n - \omega_v^n).$$

Since $\int_X (v-u)\omega_u^k \wedge \omega_v^{n-k} \geq \int_X (v-u)\omega_v^n$, we also have

$$\int_X (v-u)\omega_u^n + n\int_X (v-u)\omega_v^n \leq 0.$$

From this we get

$$-\int_X (v-u)\omega_v^n \geq \frac{1}{n+1}\int_X (v-u)(\omega_u^n - \omega_v^n).$$

Convexity of I^ρ

Let u_t be a geodesic segment connecting u_0, u_1 in $\mathcal{H}$, the space of Kähler potentials of ω. By differentiating twice we see that

- $t \mapsto \mathrm{I}(u_t)$ is affine
- $t \mapsto \mathrm{I}^\rho(u_t)$ is convex if $\rho \geq 0$ is semi-positive

Lemma 2

Assume ρ is Kähler and u_t is a geodesic segment in $\mathcal{E}^1$ such that $\omega_{u_t}^n$ is absolutely continuous with respect to ω^n. If $\mathrm{I}^\rho(u_t)$ is affine in t then $u_0 = u_1$.

Strict convexity of I^ρ

For a Kähler form $\beta \leq \rho$, I^β is also affine along u_t. By approximation, for any $\psi \in \mathcal{E}^1$, I^{ω_ψ} is also affine along u_t. We have, for a.e. $s \in [0,1]$.

$$0 = \mathrm{I}(u_1) - \mathrm{I}(u_0) = \int_X \dot{u}_s^- \omega_{u_s}^n = \int_X \dot{u}_s^+ \omega_{u_s}^n.$$

Strict convexity of I^{ρ}

Fix such s. By convexity of $\mathrm{I}^{\omega_{u_s}}$, we have

$$-\frac{1}{n}\frac{u_{s+h}-u_s}{h}\omega_{u_{s+h}}^n = \frac{\mathrm{I}^{\omega_{u_s}}(u_{s+h})-\mathrm{I}^{\omega_{u_s}}(u_s)}{h} \leq \frac{u_{s+h}-u_s}{h}\omega_{u_s}^n,$$

and

$$\int_X \frac{u_{s+h}-u_s}{h}\omega_{u_{s+h}}^n \leq \int_X \frac{u_{s+h}-u_s}{h}\omega_{u_s}^n.$$

Letting $h \to +\infty$ we see that I^{ρ} is constant along u_t.

Strict convexity of I^ρ

We then have

$$0 = (n+1)(\mathrm{I}(u_1) - \mathrm{I}(u_s)) - n(\mathrm{I}^{\omega_{u_s}}(u_1) - \mathrm{I}^{\omega_{u_s}}(u_s))$$

$$= \sum_{k=0}^{n} \int_X (u_1 - u_s)\omega_{u_1}^k \wedge \omega_{u_s}^{n-k} - \sum_{k=0}^{n-1} \int_X (u_1 - u_s)\omega_{u_1}^k \wedge \omega_{u_s}^{n-k-1} \wedge \omega_{u_s}$$

$$= \int_X (u_1 - u_s)\omega_{u_1}^n.$$

Strict convexity of I^ρ

Since

$$\int_X (u_1 - u_s)\omega_{u_1}^n \leq \int_X (u_1 - u_s)\omega_{u_1}^k \wedge \omega_{u_s}^{n-k} \leq \int_X (u_1 - u_s)\omega_{u_s}^n,$$

we infer by summing up these $(n+1)$ terms that

$$\int_X (u_1 - u_s)\omega_{u_1}^n = \int_X (u_1 - u_s)\omega_{u_s}^n = 0,$$

hence $\mathcal{I}(u_s, u_1) = 0$, which finally gives $u_s = u_1$.

Minimizer of I^{ρ} over $\mathcal{M}^1$

Recall that ω is cscK. Let G denote the identity component of the automorphism group of X.
Let $\mathcal{M}^1$ denote the set of all $\mathcal{E}^1$ minimizers φ of $\mathcal{M}$, normalized by $\mathrm{I}(\varphi) = 0$.

Lemma 3

If u is a smooth Kähler potential of ω, then there is a unique $\varphi_u \in \mathcal{M}^1$, minimizer of I^{ω_u} over $\mathcal{M}^1$ along with the following estimate

$$\mathcal{I}(\varphi_u, u) \leq n(n+1)\mathcal{I}(u, v), \;\; v \in \mathcal{M}^1.$$

Minimizer of I^{ρ} over $\mathcal{M}^1$

For each $\lambda > 0$, let φ_λ be the unique $\mathcal{E}^1$ minimizer of $\mathcal{M} + \lambda \mathrm{I}^{\omega_u}$. Fixing a $\mathcal{E}^1$-minimizer v of $\mathcal{M}$,

$$\mathcal{M}(\varphi_\lambda) + \lambda \mathrm{I}^{\omega_u}(\varphi_\lambda) \leq \mathcal{M}(v) + \lambda \mathrm{I}^{\omega_u}(v) \leq \mathcal{M}(\varphi_\lambda) + \lambda \mathrm{I}^{\omega_u}(v).$$

Hence $\mathrm{I}^{\omega_u}(\varphi_\lambda) \leq \mathrm{I}^{\omega_u}(v)$. Subtracting by $\mathrm{I}^{\omega_u}(u)$ and using Lemma 1 we obtain

$$\mathcal{I}(\varphi_\lambda, u) \leq (n+1)\mathcal{I}(u, v).$$

As $\lambda \to 0^+$ we have $\varphi_\lambda \to \varphi_u \in \mathcal{M}^1$ with

$$\mathcal{I}(\varphi_u, u) \leq (n+1)\mathcal{I}(u, v).$$

Regularity of minimizers of I^{ω_u}

Theorem 2

If $\varphi_0 \in \mathcal{M}^1$ minimizes I^{ω_u} over $\mathcal{M}^1$, then $\varphi_0 = g.0$, for some $g \in G$.

Proof.

Recall that the unique minimizer φ_λ of $\mathcal{M} + \lambda \mathrm{I}^{\omega_u}$ converges to φ_0.
Let $g \in G$ be such that $\mathrm{I}^{\omega_u}(g.0) = \min_{f \in G} \mathrm{I}^{\omega_u}(f.0)$.
We can find h smooth real function such that

$$\frac{d^2}{d\lambda^2}\Big|_{\lambda=0} (\mathcal{M} + \lambda \mathrm{I}^{\omega_u})(g.0 + \lambda h) = 0.$$

Let $\varphi_{\lambda,t}$ be the psh geodesic connecting φ_λ to $g.0 + \lambda h$. □

Minimizer of I^ρ over $\mathcal{M}^1$

- Since $\varphi_{\lambda,0}$ minimizes $\mathcal{M} + \lambda \mathrm{I}^{\omega_u}$, we have

$$O(\lambda^2) \geq \left(\frac{d}{dt}\Big|_{t=1} - \frac{d}{dt}\Big|_{t=0} \right) (\mathcal{M} + \lambda \mathrm{I}^{\omega_u})(\varphi_{\lambda,t}) \\ \geq \lambda \left(\frac{d}{dt}\Big|_{t=1} - \frac{d}{dt}\Big|_{t=0} \right) \mathrm{I}^{\omega_u}(\varphi_{\lambda,t}).$$

- Letting $\lambda \to 0$ we see that I^{ω_u} is affine along the geodesic connecting φ_0 to $g.0$, hence $\varphi_0 = g.0$.

Regularity of $\mathcal{E}^1$-minimizers of $\mathcal{M}$

We now prove Theorem 1. Assume ω is cscK and $u \in \mathcal{E}^1$ is a minimizer of $\mathcal{M}$. We want to prove that u is smooth. Let u_j be smooth Kähler potentials of ω, d_1 converging to u.
Let $\varphi_{u_j} = g_j.0$ be the unique minimizer of $\mathrm{I}^{\omega_{u_j}}$ over $\mathcal{M}^1$. Then

$$\mathcal{I}(u_j, g_j.0) \leq (n+1)\mathcal{I}(u_j, u).$$

Since $\mathcal{I}$ satisfies the quasi-triangle inequality, we infer

$$\mathcal{I}(u, g_j.0) \leq (n+1+C)\mathcal{I}(u_j, u).$$

Can extract to get $g_j \to g$, hence $u = g.0$.