Weighted cscK metrics and Mabuchi energy

I. Motivation / Definitions

(X, L) sm. cpt polarized mfld, L ample $T \subseteq \mathrm{Aut}_{red}(X)$ $[\omega_0] = 2\pi c_1(L)$

$$\mathcal{K}_T(X, \omega_0) := \{\varphi \in C^\infty(X) \mid \omega_\varphi := \omega_0 + dd^c\varphi > 0\} \cap C^\infty_T(X)$$

$$= \{T\text{-inv. Herm metrics on } L \text{ w/ pos. curv. form}\}$$

$\varphi \longmapsto h_\varphi := e^{-2\varphi} h$, h Herm metric w/ curv. form ω_0.

$$\omega^{[n]} := \frac{\omega^n}{n!}$$

There's an L^2-inner prod on $C^\infty(X, kL)$:

$$\langle s, s' \rangle_{k\varphi, v} := k^n \int_X h^k_\varphi(s, s') v(\mu_\varphi) \omega_\varphi^{[n]}.$$

Orthog. proj. $\Pi^{k\varphi}_v : L^2(X, kL) \to H^0(X, kL)$ wrt $\langle \cdot, \cdot \rangle_{k\varphi, v}$

Bergman kernel $B_w(v, k\varphi) :=$ Schwartz kernel of $w(k^{-1}A^{(k)}_\xi)\Pi^{k\varphi}_v$ res. to $\Delta \subseteq X \times X$, i.e.

$$w(k^{-1}A^{(k)}_\xi)\Pi^{k\varphi}_v(s)(x) = \int_X B_w(v, k\varphi)(x, y)\, s(y) v(\mu_\varphi) \omega_\varphi^{[n]}$$

$$B_w(v, k\varphi) = B_w(v, k\varphi)|_{\{x=y\}} \overset{\{s_i\}_{i=1}^{N_k} \text{ o.n.}}{=} v(\mu_\varphi) \sum_{i=1}^{N_k} h^k_\varphi(w(k^{-1}A^{(k)}_\xi) s_i, s_i)$$

Thm (Catlin, Ruan, Tian, Zelditch)

$$(2\pi)^n B(k\varphi) = 1 + \mathrm{Scal}(\varphi)\frac{1}{4k} + O\left(\frac{1}{k^2}\right).$$

Weighted Bergman kernel: Choose a basis $\xi_1, \cdots, \xi_\ell$ for $\mathfrak{t} = \mathrm{Lie}(T)$

Moment map $\mu_0 : X \to \mathfrak{t}^* \cong \mathbb{R}^\ell$, i.e. $d\mu_0^{\xi_i} := d\langle \mu_0, \xi_i \rangle = -\omega_0(\xi_i, \cdot)$

$$\mu_0 = (\mu_0^{\xi_1}, \cdots, \mu_0^{\xi_\ell}).$$

$$\mu_\varphi := \mu_0 + d^c\varphi$$

Moment polytope $\Delta := \mu_0(X)$ Fact, Δ is indep of φ

Weights are sm. fcns $v, w : \Delta \to \mathbb{R}$, $v > 0$

For $1 \le j \le \ell$, $A_j^{(k)} := -\sqrt{-1}\,\mathcal{L}_{\xi_j}$. $k^{-1}A_j^{(k)}$ Herm. wrt $\langle \cdot, \cdot \rangle_{k\varphi, v}$.

$\rightsquigarrow$ Spectrum of $k^{-1}A_j^{(k)}$ is $\{\lambda_i^{(k)}(\xi_j)\}_{i=1}^{N_k}$

Fact $\{\lambda_i^{(k)}\}_{i=1}^{N_k} \subset \Delta$

$$H^0(X, kL) = \bigoplus_{\lambda_i^{(k)}} \mathcal{H}(\lambda_i^{(k)}) \qquad k^{-1}A_\xi^{(k)} := (A_1^{(k)}, \cdots, A_\ell^{(k)})$$

$w(k^{-1}A_\xi^{(k)}) : H^0(X, kL) \to H^0(X, kL)$ is def'd as:

$$w(k^{-1}A_\xi^{(k)})|_{\mathcal{H}(\lambda_i^{(k)})} = w(\lambda_i^{(k)})\, \mathrm{Id}_{\mathcal{H}(\lambda_i^{(k)})}$$

Thm [Lahdili]

$$(2\pi)^n B_w(v, k\varphi) = \begin{cases} w(\mu_\varphi) + O(\tfrac{1}{k}) \\ v(\mu_\varphi) + \mathrm{Scal}_v(\varphi)\tfrac{1}{4k} + O(\tfrac{1}{k^2}) & \text{if } v = w \end{cases}$$

Def $\mathrm{Scal}_v(\varphi) := v(\mu_\varphi)\,\mathrm{Scal}(\varphi) + \underbrace{2\Delta_\varphi(v(\mu_\varphi))}_{= \Lambda_\varphi(\underbrace{-dd^c \log v(\mu_\varphi)}_{\text{curv. form for } (X \times \mathbb{C},\, v(\mu_\varphi)|\cdot|_{\mathbb{C}})})} + \underbrace{\sum_{1 \le i,j \le \ell} v_{,ij}(\mu_\varphi)\langle \xi_i, \xi_j \rangle_\varphi}_{= \langle g_\varphi,\, \mu_\varphi^*(\mathrm{Hess}(v)) \rangle}$

General setup

X cpt sm. Kähler. α fixed Kähler class $\omega_0 \in [\alpha]$

T, $\mathcal{K}_T(X, \omega_0)$, μ_φ, $\Delta \subseteq \mathfrak{t}^*$ as before

Def / Lem
- $\mathrm{Vol}_v(X, \alpha) := \int_X v(\mu_\varphi)\, \omega_\varphi^{[n]}$
- $\mathrm{Scal}_v(X, \alpha) := \int_X \mathrm{Scal}_v(\varphi)\, \omega_\varphi^{[n]}$
- θ T-inv closed $(1,1)$-form, e.g. $\mathrm{Ric}(\omega)$

  $\mathrm{Vol}^{\theta}_{v, \mathrm{Ric}}(X, \alpha) := \int_X v(\mu_\varphi)\,\theta \wedge \omega_\varphi^{[n-1]} + \langle (dv)(\mu_\varphi), \mu_\theta \rangle \omega_\varphi^{[n]}$.

are constants.

Def $\omega_\varphi \in \alpha$ T-inv Kähler metric is a (v,w)-cscK metric if

$$\mathrm{Scal}_v(\varphi) = c_{v,w}(\alpha)\, w(\mu_\varphi), \text{ where}$$

$$c_{v,w}(\alpha) = \begin{cases} \dfrac{\mathrm{Scal}_v(X,\alpha)}{\mathrm{vol}_w(X,\alpha)} & \text{, if } \mathrm{vol}_w(X,\alpha) \neq 0 \\ 1 & \text{, o/w} \end{cases}$$

II. Weighted Mabuchi energy

$$T_\varphi \mathcal{K}_T(X,\omega) = C^\infty(X,\mathbb{R})^T.$$

Def (v,w)-Mabuchi energy $\mathcal{M}_{v,w} : \mathcal{K}_T(X,\omega) \to \mathbb{R}$ is def'd by

$$\begin{cases} (d\mathcal{M}_{v,w})_\varphi(\dot\varphi) = -\int_X \dot\varphi\,(\mathrm{Scal}_v(\varphi) - c_{v,w}(\alpha)\, w(\mu_\varphi))\,\omega_\varphi^{[n]} \\ \mathcal{M}_{v,w}(0) = 0 \end{cases}$$

Thm [Lahdili] (Chen–Tian formula)

$$\mathcal{M}_{v,w} = \mathcal{H}_v - 2\mathcal{E}_v^{Ric(\omega)} + c_{v,w}(\alpha)\,\mathcal{E}_w$$

Def/Lem: i) $\begin{cases} (d\mathcal{E}_w)_\varphi(\dot\varphi) = \int_X \dot\varphi\, w(\mu_\varphi)\,\omega_\varphi^{[n]} \\ \mathcal{E}_w(0) = 0 \end{cases}$

ii) θ = T-inv. closed $(1,1)$-form

$$\begin{cases} (d\mathcal{E}_v^\theta)_\varphi(\dot\varphi) = \int_X \dot\varphi\,[v(\mu_\varphi)\,\theta\wedge\omega_\varphi^{[n-1]} + \langle (dv)(\mu_\varphi), \mu_\theta\rangle\,\omega_\varphi^{[n]}] \\ \mathcal{E}_v^\theta(0) = 0 \end{cases}$$

iii) $\mathcal{H}_v(\varphi) := \int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right) v(\mu_\varphi)\,\omega_\varphi^{[n]}$

Rmk: $\mathcal{H}_v(\varphi) = \mathrm{Ent}_{\omega^{[n]}}(v(\mu_\varphi)\,\omega_\varphi^{[n]}) + c(\alpha, v)$

Thm [Lahdili] ω is (v,w)-cscK $\Rightarrow$ ω is a global minima of $\mathcal{M}_{v,w}$.

Prop [Lahdili] If (X,α,T) admits a (v,w)-extremal metric, then id. comp. of the subgp of $\mathrm{Aut}_{red}(X)^T$ is reductive.

Thm [Lahdili] (v,w)-extremal metric is unique up to $\mathrm{Aut}^T_{red}(X)$

III. Examples

1) cscK metrics: $v = w \equiv 1$

2) v-solitons:

Def: $\omega \in 2\pi c_1(X)$ T-inv. Kähler metric is a v-soliton if

$$(*) \quad \underset{\mathrm{Ric}(\omega)}{\overset{\shortparallel}{\rho_\omega}} - \omega = \frac{1}{2} dd^c \log(v(\mu_\varphi))$$

<u>Prop 1</u> [AJL] ω_φ is v-soliton $\Leftrightarrow$ ω_φ (v,w)-cscK w/ $w = 2v(\mu_\varphi)(n + \langle d\log v(\mu_\varphi), \mu_\varphi\rangle)$

Pf "$\Rightarrow$" Taking the trace in $(*)$:

$$\mathrm{Scal}(\omega) - 2n = -\Delta_\omega(\log(v(\mu_\omega))) \quad ①$$

Contraction with ξ:

$$-d\Delta_\omega \mu_\omega^\xi + 2d\mu_\omega^\xi = d(d^c(\log v(\mu_\omega))(\xi)) \quad ②$$

Compare ① & ② $\Rightarrow$ $\mathrm{Scal}_v(\omega) = w(\mu_\omega)$

"$\Leftarrow$" Let h be a Ricci potential

Same computation $\Rightarrow$

$$0 = \mathrm{Scal}_v(\omega) - w(\mu_\omega) = v(\mu_\omega)\Delta_{\omega,v}(\log v(\mu_\omega) - h) \qquad \square.$$

Take $v = (\ell)^{-(n+2)}$, $\ell(\mu) := \langle \xi, \mu\rangle + a > 0$

<u>Prop 2</u> [AJL] TFAE: i) ω v-soliton

Apostolov – Calderbank ii) ω is $(\ell^{-(n+1)}, 2na\,\ell^{-(n+2)})$-cscK metric

iii) (N, T, ξ) is Sasaki-Einstein with transverse scalar curv. $2na$, where

$N = S^1$-bundle over $X \subseteq K_X$ with Herm metric on K_X having curv. form $-\omega$

ξ = lift of $d\ell$.

3) Extremal metrics: $v = 1$, w = affine linear.

<u>Recall</u>: ω_φ extremal iff grad $\mathrm{Scal}(\varphi)$ holo. vec. field

iff $\mathrm{Scal}(\varphi) = \mu_\varphi^\xi + c_\varphi$

<u>Moment map picture</u>:

$\mathcal{AC}_\omega^T := \{T\text{-inv. } \omega\text{-comp. almost cx structures}\} \circlearrowleft \mathrm{Ham}^T(X,\omega)$

Kähler structure on $\mathcal{AC}_\omega^T$:

$\mathrm{Lie}(\mathrm{Ham}^T(X,\omega)) \cong C^\infty(X,\mathbb{R})^T/\mathbb{R}$.

$$\Omega^{v}_{J}(\dot{J}_1,\dot{J}_2) := \frac{1}{2}\int_X \mathrm{Tr}(J\dot{J}_1\dot{J}_2)\, v(\mu_\omega)\,\omega^{[n]}$$

$$\mathbb{J}_J(\dot{J}) := J\dot{J}$$

Scalar prod on $C^\infty(X)$: $\langle \varphi,\psi\rangle_w := \int \varphi\psi\, w(\mu_\omega)\,\omega^{[n]}$.

Thm (Donaldson) (Lahdili) The moment map for $\mathrm{Ham}^T(X,\omega) \curvearrowright \mathcal{AC}^T_\omega$ is given by $\mathrm{Scal}(g(\cdot, J\cdot))$ $\left(\frac{\mathrm{Scal}_v(g_J)}{w(\mu_\omega)} - c_{v,w}(\alpha)\right)$.

Upshot Orthog. proj. of $\mathrm{Scal}(g_J)$ $\left(\frac{\mathrm{Scal}_v(g_J)}{w(\mu_\omega)}\right)$ onto $\{\mu^\xi_\varphi + c \mid \xi\in\mathfrak{t}, c\in\mathbb{R}\}$ $\cong$ $\{\langle\xi,\cdot\rangle + c \text{ on } \Delta\}$.

wrt. $\langle\cdot,\cdot\rangle$ is indep. of g_J

$\Rightarrow$ affine linear fcn $w_{ext} = \langle \xi_{ext},\cdot\rangle + c_{ext}$ s.t.

[Futaki–Mabuchi] $\mathrm{Scal}_v(\varphi) = w_{ext}(\mu_\varphi)\, w(\mu_\omega)$.

ω is extremal iff ω is $(1, w_{ext})$-cscK.

4) (v,w)-extremal metrics $(w>0)$:

Def. ω is (v,w)-extremal if $\mathrm{Scal}_v(\omega) = w(\mu_\omega)(\mu^\xi_\omega + c)$.

Prop ω is (v,w)-extremal iff ω $(v, w_{ext}w)$-cscK