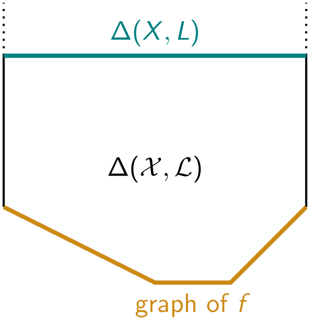
# Group actions and the effective YTD conjecture

KMS Spring meeting 2022
Special session
Group actions on varieties





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# Introduction

## Central problem in Kähler geometry

Find and study "good" Kähler metrics on a (compact) Kähler manifold $X$

Historical major results:

## Riemann Uniformization Theorem

Each compact complex curve admits a metric with constant curvature.

## Calabi-Yau Theorem

Yau's solution to Calabi conjecture $\Rightarrow$ every compact Kähler manifold with first Chern class $c_1(X) \leq 0$ admits a Kähler metric with constant Ricci curvature.

# Extremal Kähler metrics

## Definition [Calabi 1982]

A Kähler metric in a given Kähler class $\alpha$ is *extremal* if it is a minimizer of the $L^2$ norm of the scalar curvature :

$$\omega \in \alpha \mapsto \int_X S(\omega)^2 \omega^n \in \mathbb{R}$$

Scalar curvature function $S(\omega) : X \to \mathbb{R}$ defined in local coordinates by

$$S(\omega) = \frac{-n\,\partial\bar{\partial} \ln \frac{\omega^n}{idz \wedge d\bar{z}} \wedge \omega^{n-1}}{\omega^n}$$

## Theorem [Calabi 1982-1985]

- $\omega$ is extremal iff $S(\omega)$ is the potential function of a holomorphic vector field. In particular a cscK metric ($S(\omega)$ constant) is extremal.
- If $\omega$ extremal then $\mathrm{Isom}(\omega)$ is a maximal compact subgroup of $\mathrm{Aut}(X)$.

# Yau-Tian-Donaldson conjecture

Inspired by GIT stability and Kobayashi-Hitchin correspondence:

**YTD conjecture**

Existence of an extremal Kähler metric on $(X, c_1(L))$ is equivalent to an algebro-geometric condition of K-stability.

Informal definition of K-stability:
To $(X, L)$ associate *test configurations* $(\mathcal{X}, \mathcal{L})$, (degenerations of $X$ with some added conditions, see next slide).
To each test configuration $(\mathcal{X}, \mathcal{L})$ associate a number $DF(\mathcal{X}, \mathcal{L})$.

**K-stability**

$G \curvearrowleft (X, L)$ is

- ($G$-equivariantly) K-stable if $DF(\mathcal{X}, \mathcal{L}) \geq 0$ for all ($G$-equivariant) test configurations except those arising from a $\mathbb{C}^*$-action on $X$
- $G$-uniformly K-stable if $DF(\mathcal{X}, \mathcal{L}) \geq \varepsilon \|(\mathcal{X}, \mathcal{L})\|$ for a certain norm on $G$-equivariant test configurations

# Test configurations

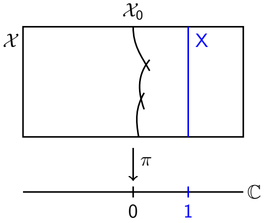
A $G$-equivariant test configuration for $G \curvearrowleft (X, L)$ consists of the data of

1. a normal $G \times \mathbb{C}^*$-variety $\mathcal{X}$,
2. a flat, projective, $\mathbb{C}^*$-equivariant morphism $\pi : \mathcal{X} \to \mathbb{C}$,
3. a $\pi$-ample line bundle $\mathcal{L}$ on $\mathcal{X}$,

such that

- $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L^r)$ for some $r \in \mathbb{Z}_{>0}$,

where $(\mathcal{X}_1, \mathcal{L}_1)$ denotes the (scheme-theoretic) fiber of $\pi$ above $1 \in \mathbb{C}$, equipped with the restriction of $\mathcal{L}$.

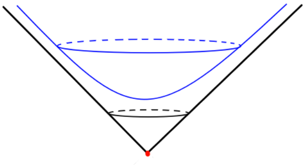


The central fiber $(\mathcal{X}_0, \mathcal{L}_0)$ has more symmetries than $X$ (equipped with an additional action of $\mathbb{C}^*$), may acquire singularities, e.g. non-reduced, several irreducible components, other singularities.

## A family of examples

Degeneration of a quadric to the cone over a lower-dimensional quadric.

- $\mathbb{P}^1$ degenerates to two intersecting lines (several irreducible components) $\mathcal{X} = \{([x:y:z], t); xy - tz^2 = 0\}$
- $\mathbb{P}^1$ degenerates to a double line (non-reduced) $\mathcal{X} = \{([x:y:z], t); txy - z^2 = 0\}$
- $\mathbb{P}^1 \times \mathbb{P}^1$ degenerates to a weighted projective space (normal, but singular)



# Panorama of (some) key results

[Futaki 1983] obstruction when degeneration to $X$ itself induced by $\mathbb{C}^*$ action

[Ding-Tian 1992] obstruction from degenerations to smooth manifolds

[Wang-Zhu 2004] Fano toric manifolds, non-existence KE $\Leftrightarrow$ Futaki's obstruction

[Donaldson 2009] YTD conjecture for cscK metrics on toric surfaces

[Chen-Donaldson-Sun, Tian, 2015] YTD for Fano Kähler-Einstein metrics

[Berman-Darvas-Lu 2020] existence extremal $\Rightarrow$ uniform K-stability

[Chen-Cheng 2021] From the analytical point of view, proved that coercivity (modulo automorphisms) of the Mabuchi functional implies existence of cscK metrics

[Chi Li 2021] some algebraic notion close to uniform K-stability $\Rightarrow$ existence cscK

# Towards a more effective version of K-stability?

## Effective YTD conjecture

To check (uniform) K-stability of $(X, L)$, it is enough to consider test configurations whose central fiber has at most $\dim(X)$ irreducible components.

## Example

- [Donaldson 02] cscK on toric surfaces, 2 irred comp are enough (but not 1)
- [Wang-Zhu 2004] $(X, K_X^{-1})$ toric Fano manifold, 1 irred comp is enough
- [Apostolov,Calderbank,Gauduchon,Tonnesen-Friedman 2008] 2 irred comp are enough for certain "admissible" $\mathbb{P}^1$-bundles (but not 1)
- [Li-Xu 2011] $(X, K_X^{-1})$ Fano, 1 irred comp is enough
- [D. 2020] for cohomogeneity one manifolds, 1 is enough

**Upshot:**

- under additional conditions can hope for finite dimensional space of conditions
- in Fano case, basis upon which the delta invariant and subsequent Abban-Zhuang strategy were built.

# Manifolds with large symmetry: spherical varieties

## Definition

Let $G$ complex connected linear reductive group. A normal $G$-variety $X$ is spherical if a Borel subgroup $B$ of $G$ acts with an open dense orbit in $X$.

**Upshot:** open $G$-orbit + **moment polytope** classify spherical varieties [Losev]

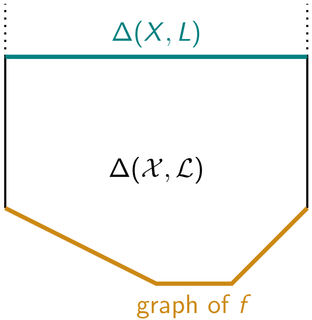
There is also a combinatorial classification of possible open orbits.
Important for us: **valuation cone**, inside dual of direction of moment polytope.

## Theorem [D. 2020 and appendix by Odaka]

- $G$-spherical manifold $(X, L)$ admits a cscK metric iff $G$-uniformly K-stable.
- convex geometric translation on the moment polytope of $G$-uniform K-stability.

## Theorem [D. 2020]

- $G$-equivariant test configurations of $(X, L)$ are in 1:1 correspondence with negative rational piecewise linear convex functions on the moment polytope $\Delta$, whose slopes are in the opposite valuation cone $-\mathcal{V}$ of $X$.
- irreducible components of central fiber correspond to linearity domains of that function. In particular, for such varieties, the space of test configurations whose central fiber has $\leq \dim(X)$ components is finite dimensional.



Set $\Phi_X^+ = \{\alpha \in \Phi^+(G) \mid \alpha|_\Delta \not\equiv 0\}$ and $\varpi_X = \sum_{\alpha \in \Phi_X^+} \alpha$

Let $P(\bullet) = \prod_{\alpha \in \Phi_X^+} \frac{\langle \bullet, \alpha \rangle}{\langle \varpi_X, \alpha \rangle}$ and $Q(\bullet) = \sum_{\alpha \in \Phi_X^+} \frac{\langle \varpi_X, \alpha \rangle}{\langle \bullet, \alpha \rangle} P(\bullet)$

## Uniform K-stability criterion [D. 2020]

$(X, L)$ is $G$-uniformly K-stable if and only if there exists $\varepsilon > 0$ such that for all convex PL function $f$ on $\Delta$ with slopes in $-\mathcal{V}$,

$$\int_{\partial\Delta} fPd\sigma + \int_\Delta f2(Q - aP)d\mu \geq \varepsilon \inf_{l \in \mathrm{Lin}(\mathcal{V})} \int_\Delta (f + l - \min(f + l))Pd\mu$$

# Applications

## Theorem [D. 2020]

A **rank 1** polarized $G$-spherical manifold $(X, L)$ admits a cscK metric if and only if it is K-stable with respect to $G$-equivariant test configurations with an irreducible central fiber.
The latter translates into a very simple single combinatorial condition $\sim$ sign of a polynomial evaluated at one single point.

Rank 2: was mentionned in Yan Li's talk

## Theorem [D.2020]

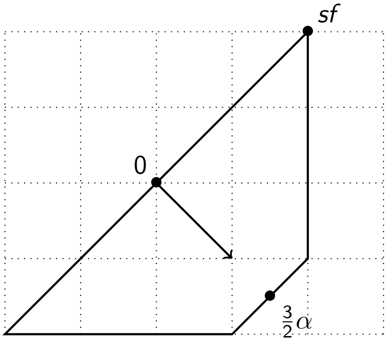
Combinatorial sufficient condition for uniform K-stability of spherical varieties.
Applies particularly well for close to Fano spherical manifolds.

# Examples

Consider the $\mathrm{SL}_2 \times \mathbb{C}^*$-spherical variety $\mathrm{Bl}_{Q^1} Q^3$. Let $\alpha$ be the unique positive root and $f$ generating character of $\mathbb{C}^*$. Up to scaling, the moment polytope of an ample line bundle is as on the right

By the sufficient condition, the associated Kähler class admits a cscK metric if

$$\frac{1683}{1000} < s < 3$$




## Theorem [D. 2019]

If $2 \leq k \leq n-3$, $(X = \mathrm{Bl}_{Q^k} Q^n, K_X^{-1})$ is K-unstable and does not admit any Kähler-Ricci soliton.

# Hidden symmetries : fiber bundles

- $(X, \omega_X)$ $T$-toric manifold with moment polytope $\Delta$
- $B = \prod(B_a, \omega_a)$ product of cscK Hodge manifolds
- $(Q, \theta)$ principal $T$-bundle with connection and $d\theta = \sum_a p_a \otimes \omega_a$

**Semisimple principal toric bundle:** Kähler manifold $(Y, \omega_Y)$ where

$$Y = Q \times X/T \qquad \text{and} \qquad \omega_Y = \omega_X + \sum_a c_a \omega_a + d\langle \mu, \theta \rangle$$

## Theorem [Jubert 2021]

- A uniform YTD conjecture holds for semisimple principal toric bundles
- (analogue of Matsushima's theorem) An extremal metric must behave well with respect to the bundle structure

# Applications

**Theorem [D.-Jubert 2022]**

Simple combinatorial sufficient condition when the fiber is a Fano toric manifold, can run a computer program to check existence of extremal Kähler metric.

**Example**

- $X = \mathbb{P}^2$
- $B$ Kähler-Einstein Fano threefold, $H$ is the smallest integral divisor of $c_1(B)$
- $Y = \mathbb{P}_B(O \oplus H^{-p_1} \oplus H^{-p_2})$ with $1 \leq p_1 \leq p_2$

There exists extremal Kähler metric in Kähler class $c_1(X) + \lambda c_1(B)$ for $\lambda \geq 7p_2$.

Here slight abuse of notations: $c_1(X)$ relative first Chern class
$c_1(B)$ identified with its pull-back